DISCUSSION PAPER SERIES

No. 10588

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INTERNATIONAL MACROECONOMICS

Centre for Economic Policy Research

ISSN 0265-8003

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Discussion Paper No. 10588
May 2015
Submitted 30 April 2015

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JEL Classification: E62, F33, F34, F53, H12 and H63
Keywords: austerity programs, debt overhang, default, European debt crisis, fiscal policy, Great Recession, Greece, International Monetary Fund, limited commitment, moral hazard, renegotiation, risk premia, sovereign debt and structural reforms

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Sovereign Debt and Structural Reforms*

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May 6, 2015

Abstract

Motivated the European debt crisis, we construct a tractable theory of sovereign debt and structural reforms under limited commitment. The government of a sovereign country which has fallen into a recession of an uncertain duration issues one-period debt and can renege on its obligations by suffering a stochastic default cost. When faced with a credible default threat, creditors can make a take-it-or-leave-it debt haircut offer to the sovereign. The risk of renegotiation is reflected in the price at which debt is sold. The sovereign government can also do structural policy reforms that speed up recovery from the recession. We characterize the competitive equilibrium and compare it with the constrained efficient allocation. The equilibrium features increasing debt, falling consumption, and a non-monotone reform effort during the recession. In contrast, the constrained optimum yields step-wise increasing consumption and step-wise decreasing reform effort. Markets for state-contingent debt alone do not restore efficiency. The constrained optimum can be implemented by a flexible assistance program enforced by an international institution that monitors the reform effort. The terms of the program are improved every time the country poses a credible threat to leave the program unilaterally without repaying the outstanding loans.

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1 Introduction

The European debt crisis has revamped the debate on sovereign debt crises. The Great Recession hit Southern European economies especially hard. Borrowing from international financial markets to smooth consumption would seem the natural response for economies in a downturn. However, some countries – most notably Greece and Italy – had entered the recession with an already large outstanding debt, in excess of 100% of their annual GDP. In 2009, a confidence crisis triggered growing default premia. As the cost of servicing debt ran rampant, it triggered social and political unrest, including pressure not to honor the outstanding financial obligations.

The parable of Greece is emblematic. As the Greek debt-to-GDP ratio climbed up from 107% in 2008 to 146% in 2010, international organizations stepped in to provide financial assistance and

*Kjetil Storesletten acknowledges support from the European Research Council (ERC Advanced Grant IPCDP-324085).

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access to new loans, asking in exchange a commitment to economic reforms. In May 2010 the Troika (a 3-part commission representing the IMF, the EU and the ECB) launched a 110 billion Euro bailout conditional on a set of austerity measures (to be followed by a similar intervention in 2012). However, austerity was met by sharp political opposition. Amidst angry street protests, pressure escalated not to honor the outstanding debt that had meanwhile increased to over 170% of GDP. In October 2011, creditors had to agree to a debt haircut implying a 53% loss on its face value. After a temporary decline, debt (now consisting largely of institutional loans) soared again. The recent electoral victory of the radical party Syriza in early 2015 calls into question both the austerity policy, and when and to what extent the outstanding debt should be repaid.

Motivated by these recent events, in this paper we construct a theory of sovereign debt aimed to address the following set of questions. First, what does the optimal dynamic contract between a planner and a sovereign temporarily impoverished country prescribe in an environment where the country cannot commit to honoring its debt? Second, what policies and institutional arrangements can implement it? In particular, does the market equilibrium attain the efficient level of structural reforms and debt dynamics? Or, otherwise, can an international institution (such as the IMF) introduce welfare-improving programs? Finally, how should this institution deal with pressure to renegotiate an existing assistance program?

The theory rests on three building blocks. The first is that sovereign debt is subject to limited enforcement, and that countries can renege on their obligations subject to real costs, as in Arellano (2008). Different from Arellano (2008), we assume the size of default costs to be stochastic, reflecting exogenous changes in the domestic and international situation. For instance, when "dovish" center-left governments took office in Italy and France, the pressure on Greece went down. The second building block is that whenever creditors face a credible default threat, they can make a take-it-or-leave-it renegotiation offer to the indebted country. This approach conforms with the empirical observations that unordered defaults are rare events, and that there is great heterogeneity in the terms at which debt is renegotiated, as documented by Tomz and Wright (2007) and Sturzenegger and Zettelmeyer (2008). The third building block is the possibility for the government of the indebted country to make "structural" policy reforms that speed up recovery from an existing recession.

More formally, we construct a dynamic stochastic general equilibrium model with an endogenous debt price. Productivity is subject to aggregate shocks following a two-state Markov process. A benevolent local government can issue sovereign debt to smooth consumption.[1] The sovereign country starts in a recession of an unknown duration. The probability that the recession ends is endogenous, and hinges on the government's reform effort. Debt issuance is subject to a limited-commitment problem: the government can, *ex-post,* repudiate its debt, based on the publicly observable realization of a stochastic default cost. When this realization is sufficiently low relative to the outstanding debt, the default threat is credible. In this case, a syndicate of creditors makes a take-it-or-leave-it debt haircut offer, as in Bulow and Rogoff (1989). In equilibrium, there is no outright default, but repeated debt renegotiations. Haircuts are more frequent during recession, and the larger the outstanding sovereign debt is.

We first characterize the laissez-faire equilibrium. During recessions, the government would like to issue debt in order to smooth consumption. However, as debt accumulates, the probability of renegotiation increases, implying a growing risk premium. The equilibrium does not feature full insurance. In a recession, consumption falls as debt accumulates. The reform effort exhibits a non-

[1] In the benchmark model, sovereign debt is a non-state-contingent bond. Our results do not hinge on this assumption, and we show in an extension that if the government can issue state-contingent debt, i.e., securities whose pay-off is contingent to the realization of the aggregate productivity, the main results remain unchanged.

monotonic pattern: it is increasing with debt at low levels of debt because of the disciplining effect of recession (the welfare cost of the recession is higher when the country must service a large debt). However, for sufficiently high debt levels the relationship is flipped. The reason is that part of the benefits of the reform accrues to creditors, and more so the higher the debt level. Thus, the theory features a version of the debt overhang problem highlighted by Krugman (1988): very high debt levels deter useful reforms. The moral hazard problem exacerbates the country's inability to achieve consumption smoothing: at high debt levels, creditors expect a low reform effort, are pessimistic about the future economic outlook, and request an even higher risk premium.

Next, to establish a normative benchmark, we characterize the optimal dynamic contract between a planner without enforcement power and a country that has fallen into a recession. In contrast with the competitive equilibrium, the constrained optimal allocation features non-decreasing consumption and non-increasing reform effort during the recession. More precisely, consumption and effort remain constant whenever the country's participation constraint is not binding. However, when the constraint is binding (corresponding to a low realization of the default cost), the planner increases the country's promised utility and consumption, and reduces its reform effort.

Having characterized the constrained-efficient allocation, we consider its implementation in a decentralized environment. We first show that, in the absence of aggregate productivity shocks, the laissez-faire equilibrium attains the constrained-efficient allocation. However, the equilibrium is not constrained-efficient when the economy is in a recession. In the laissez-faire equilibrium, there is too little consumption smoothing, and the reform effort is inefficiently low. Interestingly, the inefficiency cannot be resolved by allowing the government to issue state-contingent debt. In standard models, state-contingent debt provides insurance against the continuation of a recession – i.e., Arrow securities paying off conditional on the aggregate state, recession or normal time. However, the better insured is the country, the more severe the moral hazard problem becomes. For instance, full insurance would destroy any incentive to exert reform effort. Since creditors would anticipate the moral hazard problem, this would be priced into the debt. Thus, the social value of markets for state-contingent debt is limited. In a calibrated version of the model, we show that having access to state-contingent debt yields only small welfare gains in this environment.

While the implementation of the constrained efficient allocation requires interim montoring, which is a strong assumption, we show that a weaker concept of constrained efficient can be attained if the planner can observe the reform effort ex post, and condition the continuation of the program to the execution of the desired reform.

Although not implemented by the laissez-faire equilibrium, the constrained optimal allocation can be attained through the intervention of an independent institution (e.g., the IMF) that has the power (i) to control the country's fiscal policy (an *austerity program*); (ii) to monitor the reform effort (possibly ex post). During the recession the optimal program entails a persistent budget support by extending loans on favorable terms, combined with a specified reform effort, larger than the borrower would choose on its own. Upon recovery from the recession, the sovereign is settled with a (large) debt on market terms. A common objection to schemes implying deferred repayment is that the country may refuse to repay its loans when the economy recovers. In our theory, this risks exists, but is taken into account *ex-ante* when the deal is agreed upon. The larger the probability of future non-repayment, the harsher conditions the country must accept upon entering the assistance program. The program can in principle be budget-neutral, in expected terms, for the international institution. Ex-post, it can result in either gains or losses depending on the evolution of the crisis.

The optimal program has the interesting feature that, whenever a credible default threat is on the table, the international institution should give in and improve the terms of the agreement for the debtor by granting her higher consumption and a lower reform effort. In other words, the austerity program is relaxed over time whenever this is necessary to avert the breakdown of the program. Notably, the efficiency of the contract is not enhanced if the institution can credibly threaten to stop its financial support whenever the debtor tries to renegotiate the terms that were initially agreed upon. Intuitively, such a threat would increase the probability that the government honors its debt, but could not prevent default when its cost is very low. In the event of a default, the country would suffer a real cost, being then forced to revert to the competitive equilibrium, which is not efficient. The international institution, in turn, would lose all the resources invested in the assistance program.

These observations have interesting policy implications for the recent debate about the management of the European debt crisis. The request of Greece to renegotiate the austerity conditions has been met by fierce opposition from Germany. One of the arguments is that accepting a renegotiation would have perverse incentive consequences on the reform process in Greece. Our theory predicts that, to the extent that Syriza's threat is credible, appeasement may be the optimal response for the European Union, so long as the alternative in outright default. Interestingly, a post-default scenario may entail less structural reforms than one where the demands of Greece are appeased and default is averted.

We provide a quantitative evaluation of the theory with the aid of a calibrated version of the model. The model matches realistic debt-to-GDP ratios, as well as default premia and recovery rates. We regard this as a contribution of its own. In the existing quantitative literature, it is difficult to sustain high debt levels, contrary both to the observation that many countries have managed to finance debt-GDP ratios above 100%, and to the estimates of a recent study by Collard, Habib, and Rochet (2015) showing that OECD countries can sustain debt-GDP ratios even in excess of 200%. We find that an assistance program implementing the constrained optimum yields large welfare gains, equivalent to a transfer of 63% of the initial GDP with a zero expected cost for the institution running the assistance program.

1.1 Literature review

Our paper relates to several streams of the literature on sovereign debt. The seminal contribution to the analysis of debt repudiation in models with incomplete markets is Eaton and Gersovitz (1981). In their model, the incentive to repay is sustained by the threat of future exclusion from credit markets. Fernandez and Rosenthal (1989) study debt renegotiation in a game-theoretic framework where default penalties are not credible, and the incentive for the renegotiating country to repay takes the form of an improved access to international capital markets. Relative to these early contributions, our model provides a less detailed description of the renegotiation game. Our theory is closer to the approach of Bulow and Rogoff (1989). In their paper, a country and a bank renegotiate over time what proportion of the debt must be serviced. Renegotiation is costless, and default penalties (defined by the seizure of part of international trade) define the threat point for renegotiation. Repeated renegotiation is an equilibrium outcome, as in our model.

Our work is also closely related to the more recent models of Arellano (2008) and Yue (2010). Arellano (2008) assumes that default is subject to a real cost (e.g., trade loss), and studies how the probability of default varies with the severity of recession. Yue (2010) considers, as we do, the possibility of renegotiation, although in her model renegotiation is costly and is determined by Nash bargaining between creditors and debtors - with no stochastic shocks to outside options.[2] In her model, ex-post inefficient restructuring helps ex-ante discipline and provides incentives to honor the

[2] Other papers focusing on the restructuring of sovereign debt include Asonuma and Trebesch (2014), Benjamin and Wright (2009), Bolton and Jeanne (2007), Dovis (2014), Hopenayn and Werning (2008), and Mendoza and Yue (2012).

debt. Neither Arellano (2008) nor Yue (2010) study the efficient allocation and its implementation through an assistance program. Moreover, we pursue an analytical characterization of the properties of the model, whereas their main focus is quantitative. One problem in the quantitative literature is that the equilibrium can sustain debt levels that are much lower than what is observed in the data. Our model, by assuming a more efficient renegotiation process, can sustain higher and more realistic debt-GDP ratios.

Another recent paper complementary to ours is Conesa and Kehoe (2015). In their theory, under some circumstances, the government of the indebted country may opt to "gamble for redemption." Namely, it runs an irresponsible fiscal policy that sends the economy into the default zone if the recovery does not happen soon enough. While this is remindful of the debt overhang feature of our theory, the source and the mechanism of the crisis is different. Their model is based on the framework of Cole and Kehoe (1996, 2000) inducing multiple equilibria and sunspots. Our model features instead a unique equilibrium, due to a different assumption about the timing of default and the issuance of new debt.

Hopenhayn and Werning (2008) study the optimal contract between a bank and a risk neutral borrowing firm. Like us, they assume that the borrower has a stochastic default cost. However, they focus on the case when this outside option is not observable to the lender and show that this implies that default can occur in equilibrium. Unlike us, they do not study reform effort and they do not analyze the case of sovereign debt issued by a country in recession.

Our paper is also related to the literature on endogenous incomplete markets due to limited enforcement. This includes Alvarez and Jermann (2000) and Kehoe and Perri (2002). The analysis of constrained efficiency is also related to the literature on competitive risk sharing contracts with limited commitment, including Thomas and Worrall (1988), Kocherlakota (1996), and Krueger and Uhlig (2006). An application of this methodology to the optimal design of a Financial Stability Fund is provided by Abraham, Carceles-Poveda, and Marimon (2014). An excellent review of the literature on sovereign debt with limited enforcement can be found in Aguiar and Amador (2014).

In the large empirical literature, our paper is related to the finding of Tomz and Wright (2007). Using a dataset for the period 1820–2004, they find a negative but weak relationship between economic output in the borrowing country and default on loans from private foreign creditors. While countries default more often during recessions, there are many cases of default in good times as well as many instances in which countries have maintained debt service during times of very bad macroeconomic conditions. They argue that these findings are at odds with the existing theories of international debt. Our theory is instead consistent with the pattern they document. In our model, due to the stochastic default cost, countries may default during booms (though this is less likely, consistent with the data) and can conversely fail to renegotiate their debt during very bad times. Their findings are reinforced by Sturzenegger and Zettelmeyer (2008) who document that even within a relatively short period (1998-2005) there are very large differences between average investor losses across different episodes of debt restructuring.[3] The observation of such a large variability in outcomes is in line with our theory, insofar as the bargaining outcome hinges on an outside option that is subject to stochastic shocks. Borensztein and Panizza (2009) evaluate empirically the costs that may result from an international sovereign default, including reputation costs, international trade exclusion, costs to the domestic economy through the financial system, and political costs to the authorities. They find that the economic costs are generally short-lived. For a more thorough review of the evidence, see also Panizza, Sturzenegger, and Zettelmeyer (2009).

[3] In their data, losses range from 13 percent (Uruguay, 2003) to 73 percent (Argentina, 2005).

2 The model environment

The model economy is a small open endowment economy populated by an infinitely-lived representative agent. The endowment process follows a two-state Markov switching regime, with realizations $w \in \{\underline{w}, \bar{w}\}$. We assume the state labelled *normal times*, $\bar{w}$, to be an absorbing state. If the economy starts in a recession $(\underline{w})$, it permanently leaves the recession with probability p and remains in the recession with probability $1-p$. This assumption allows us to focus sharply on anomalous single events such as the Great Recession.

A benevolent government can issue a one-period bond (sovereign debt) to smooth consumption, and can implement a costly reform policy to increase the probability of a recovery. Once the economy is out of recession, there is no further need of reforms. In our notation, p is both the *reform effort* and the probability that the recession ends. At the beginning of each period, before issuing new debt, the government also decides whether to honor or to repudiate the outstanding debt that comes to maturity.

The preferences of the representative household are represented by the following expected utility function:

$$E_0 \sum \beta^t \left[\ln (c_t) - \phi I_{\{\text{default in } t\}} - X(p_t) \right].$$

$I \in \{0,1\}$ is an indicator variable switching on when the economy is in a default state; ϕ is a stochastic default cost assumed to be i.i.d. over time and drawn from the p.d.f. $f(\phi)$ with an associated c.d.f. $F(\phi)$. We assume that $f(\phi)$ has no mass points, and denote the support of the p.d.f. by $\aleph \subseteq \mathbb{R}^+$, where $\aleph$ is assumed to be a convex set. We denote the lower bound of $\aleph$ by $\phi_{\min} \geq 0$. The assumption that shocks are independent is inessential, but aids tractability. X is the cost of reform, assumed to be an increasing convex function of the probability of exiting recession. More formally, we assume that $p \in [\underline{p}, \bar{p}] \subseteq [0,1]$, $X(\underline{p}) = 0$, $X'(p) > 0$ and $X''(p) > 0$. In normal times, $X = 0$.

In a frictionless complete market economy, the country would obtain full consumption. As we show in Section 4.1, if $\beta R = 1$ consumption is constant throughout, and effort is constant until the recession ends.

3 Competitive equilibrium

In this section, we characterize the laissez-faire equilibrium. The only asset is a one-period bond, b. This is a claim to one unit of next-period consumption good, which sells today at the price $Q(b, w)$. The bonds are purchased by a representative foreign creditor assumed to be risk neutral and to have access to international risk-free assets paying the world interest rate R. After issuing debt, the country decides its reform effort.

The key assumptions are that (i) the country cannot commit to repaying its sovereign debt, and (ii) the reform effort is exerted after the debt is issued and is not contractible. At the beginning of each period, the government decides whether to repay the debt that comes to maturity or to announce default on all its debt. Default is subject to a stochastic cost, denoted by ϕ, capturing in a reduced form a variety of shocks (e.g., the election of a new prime minister, a new central bank governor taking office, the attitude of foreign governments, etc.). ϕ is publicly observed. If a country defaults, no debt is reimbursed. In this case, the government cannot issue new debt in the default period, but can start issuing bonds already in the following period.[4]

[4]Our model can be enriched with richer post-default dynamics, such as prolonged or stochastic exclusion from debt

After observing the realization of ϕ, creditors can make a take-it-or-leave-it renegotiation offer. By accepting the renegotiation offer, the government averts the default cost. In equilibrium, a haircut is only offered if the default threat is credible , i.e., if the realization of ϕ is sufficiently low to make the country prefer default to full repayment. When they offer renegotiation, creditors make the debtor indifferent between an outright default and the proposed haircut.

More formally, the timing is as follows: The government enters the period with the pledged debt b, then observes the realization of w and ϕ, and then decides whether to announce default. If a threat is on the table, the creditors offer a haircut. Next, the country decides whether to accept or decline the offer. Finally, the government decides its reform effort, and consumption is realized.

After repaying, in part or in full, the outstanding debt, the government sets the new debt level, b', subject to the following period budget constraint:

$$Q\left(b', w\right) \times b' = b + c - w. \tag{1}$$

In case of default, the country cannot borrow in the current period. Thus, $b' = 0$ and $c = w$. If the country could commit, it would sell bonds at the price $Q(b, w) = 1/R$. However, due to the risk of default or renegotiation, it must generally sell at a discount, $Q(b, w) \leq 1/R$.

The benevolent government's value function can be written as

$$V(b, \phi, w) = \max\left\{W^H(b, w), W^D(\phi, w), W^R(\phi, w)\right\}, \tag{2}$$

where H indicates "honor", D "default", and R "renegotiation." Thus,

$$\begin{aligned} W^H(b, w) &= \max_{b'}\left\{\ln\left(Q\left(b', w\right) \times b' + w - b\right) + Z\left(b', w\right)\right\}, \\ W^D(\phi, w) &= \max_{p}\left\{\ln(w) - \phi + Z(0, w)\right\}, \\ W^R(\phi, w) &= \max_{b'}\left\{\ln\left(Q\left(b', w\right) \times b' + w - \hat{b}(\phi, w)\right) + Z\left(b', w\right)\right\}, \end{aligned}$$

where[5]

$$\begin{aligned} Z\left(b', \underline{w}\right) &= \max_{p}\left\{-X(p) + \beta\left(p \times E\left[V\left(b', \phi', \bar{w}\right)\right] + (1-p) \times E\left[V\left(b', \phi', \underline{w}\right)\right]\right)\right\}, && (3) \\ Z\left(b', \bar{w}\right) &= \beta E\left[V\left(b', \phi', \bar{w}\right)\right], && (4) \end{aligned}$$

and $E\left[V\left(b', \phi', w\right)\right] = \int_{\aleph} V(b, \phi, w)\, dF(\phi)$. The function $\hat{b}(\phi, w)$ denotes the renegotiated debt level that makes the government indifferent between accepting the creditors' offer and defaulting, and is determined implicitly by the equation $W^H\left(\hat{b}(\phi, w), w\right) = W^D(\phi, w)$.

Since outright default is never observed in equilibrium, the value functions simplify to:

$$V(b, \phi, \underline{w}) = \max_{b'}\left\{\begin{array}{c} \ln\left(Q\left(b', \underline{w}\right) \times b' + \underline{w} - \min\left\{b, \hat{b}(\phi, \underline{w})\right\}\right) \\ + \max_p\left\{-X(p) + \beta\left(p \times E\left[V\left(b', \phi', \bar{w}\right)\right] + (1-p) \times E\left[V\left(b', \phi', \underline{w}\right)\right]\right)\right\} \end{array}\right\},$$

markets. Note that one can as well regard some of these additional costs to be captured by the shock ϕ. Since outright default does not occur in equilibrium, the details of the post-default dynamics are immaterial.

[5] Note that the reform effort is not an argument of the bond price function, $Q(b', w)$, since it is chosen after b' is issued.

$$V(b,\phi,\bar{w}) = \max_{b'} \left\{ \ln\left(Q\left(b',\bar{w}\right) \times b' + \bar{w} - \min\left\{ b, \hat{b}\left(\phi,\bar{w}\right) \right\} \right) + \beta E\left[V\left(b',\phi',\bar{w}\right) \right] \right\}.$$

In the rest of the paper, it is convenient to use the more compact notation $EV(b,w) \equiv E[V(b,\phi,w)]$ and $EV(b',w) \equiv E\left[V\left(b',\phi',w\right)\right]$.

Let $\Phi(b,w)$ define the threshold default shock realization such a government with the initial debt b can credibly threaten to default for all $\phi \leq \Phi(b,w)$. More formally, $W^H(b,w) = W^D(\Phi(b,w),w)$, implying that:[6]

$$\Phi(b,w) = \ln(w) + \max_p \left\{ -X(p) + \beta\left[pW^H(0,\bar{w}) + (1-p)W^H(0,\underline{w}) \right] \right\} - W^H(b,w).$$

The following Lemma can be established:

Lemma 1 *$\hat{b}(\Phi(b,w),w) = b$. Hence, $\hat{b}(\phi,\bar{w}) = \bar{\Phi}^{-1}(\phi)$ and $\hat{b}(\phi,\underline{w}) = \underline{\Phi}^{-1}(\phi)$, where $\bar{\Phi}(b) \equiv \Phi(b,\bar{w})$ and $\underline{\Phi}(b) \equiv \Phi(b,\underline{w})$.*

The Lemma follows from the definitions of $\hat{b}$ and Φ : recall that $\hat{b}(\phi,w)$ is the debt level that, conditional on ϕ, makes the debtor indifferent between honoring and defaulting. In turn, $\Phi(b,w)$ is the realization of ϕ that, conditional on b, makes the debtor indifferent between honoring and defaulting.

3.1 Equilibrium in normal times

In this section, we characterize the equilibrium under no aggregate uncertainty, namely, when the economy starts in normal times ($w = \bar{w}$). We start from the characterization of the equilibrium price of debt. Then, we analyze the equilibrium consumption/debt issuing dynamics.

Since creditors are risk neutral, the expected rate of return on the sovereign debt must equal the risk-free rate of return. Arbitrage implies, then, the following bond price:

$$Q(b,\bar{w}) = \frac{1}{R}\left(\left(1 - F\left(\bar{\Phi}(b)\right)\right) + \frac{1}{b}\int_0^{\bar{\Phi}(b)} \left(\bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi \right) \right). \tag{5}$$

The first term within parenthesis on the right-hand side yields the probability that debt is fully honored, in which case debt is fully recovered. The second term yields the recovery rate under renegotiation. Equation (5) allows us to characterize the endogenous debt limit. To this aim, define $\bar{b}$ as the lowest debt level inducing renegotiation almost surely (i.e., such that $\lim_{b \to \bar{b}} F\left(\bar{\Phi}(b)\right) = 1$). The next Lemma establishes that $\bar{b}$ is also the top of the Laffer curve, i.e., the endogenous debt limit.[7]

Lemma 2 *$\bar{b} = \arg\max_b \{Q(b,\bar{w})\, b\}$.*

[6] Note that $\Phi(b,w)$ is unique since W^D is decreasing whereas W^H is independent of ϕ.

[7] Note that the Lemma implies that

$$\begin{aligned} \lim_{b \to \bar{b}} Q(b) &= \frac{1}{R}\frac{1}{\bar{b}}\int_0^{\infty} \left(\Phi^{-1}(\phi,\bar{w}) \times f(\phi)\ d\phi \right) > 0 \\ &\Rightarrow \quad Q\left(\bar{b}\right)\bar{b} = \frac{1}{R}E\left\{ \Phi^{-1}(\phi,w) \right\}. \end{aligned}$$

We can now move to the consumption decisions and to the associated debt dynamics. We introduce a definition that will be useful throughout the paper.

Definition 1 *A Conditional Euler Equation (CEE) is the equation describing the (expected) ratio of the marginal utility of consumption in all states of nature such that ϕ' induces the government to honor its debt.*

Next, we characterize formally the CEE.[8] The sovereign government solves the following problem:

$$B\left(\tilde{b}, \bar{w}\right) = \arg\max_{b'} \left\{ \ln\left(Q\left(b', \bar{w}\right) \times b' + \bar{w} - \tilde{b}\right) + \beta E V\left(b', \bar{w}\right) \right\}, \tag{8}$$

where $\tilde{b} = \min\left\{b, \hat{b}\left(\phi, \bar{w}\right)\right\}$.

Proposition 1 *Suppose that the government issues the debt level b'. Then, if the realization of ϕ' induces no renegotiation, the following CEE holds true:*

$$\beta R \frac{c}{c'|_{H,\bar{w}}} = 1, \tag{9}$$

where

$$c = C\left(b', w; \tilde{b}\right) = Q\left(b', w\right) \times b' + w - \tilde{b} \tag{10}$$

is current consumption, and

$$c'|_{H,w} = C^H\left(b', w\right) = Q\left(B\left(b', w\right), w\right) \times B\left(b', w\right) + w - b' \tag{11}$$

is next-period consumption conditional on no renegotiation.

The CEE (9) states that, in all states in which debt is fully honored, consumption growth equals βR. Although the CEE resembles a standard Euler equation under full commitment, the similarity is deceiving: R is *not* the ex-post interest rate when debt is fully honored – the realized interest rate is in fact higher due to the default premium.

When debt is renegotiated, debt falls discretely and consumption jumps up. Thus, consumption growth exceeds βR. That consumption growth is higher under renegotiation is not surprising, since in this case the country benefits from a reduction in the repayment to creditors.

Henceforth, we simplify the analysis by assuming that $\beta R = 1$. In this case, consumption and debt remain constant in every period in which the country honors its debt, while consumption increases discretely upon every episode of renegotiation. In the webpage appendix, we characterize the general case where $\beta R \neq 1$.

[8] In the appendix we prove that, when $\beta R = 1$, the default threshold $\hat{b}\left(\phi, \bar{w}\right)$ has the following expression:

$$\hat{b}\left(\phi, \bar{w}\right) = \frac{\beta}{1 - \beta\left(1 - F\left(\phi\right)\right)} \int_0^\phi \hat{b}\left(x, \bar{w}\right) f\left(x\right) dx \tag{6}$$

$$+\bar{w} \frac{1 - \exp\left(\beta \int_0^\phi x f\left(x\right) dx - \left(1 - \beta\left(1 - F\left(\phi\right)\right)\right) \times \phi\right)}{1 - \beta\left(1 - F\left(\phi\right)\right)} \tag{7}$$

See Lemma 7 and the ensuing proof.

3.1.1 Taking stock

In normal times, the only source of uncertainty is the realization of the default cost. If $\phi_{\min} > 0$, an economy with a sufficiently low debt level never experiences debt crises. In general, however, debt crises and renegotiations happen recurrently.

Figure 1 shows the consumption and debt dynamics for $\beta R = 1$. Consumption and debt remain constant in every period in which the country honors its debt. In contrast, consumption increases step-wise every time the debt is renegotiated down. Eventually, a sequence of renegotiations brings the debt to a sufficiently low level where the risk of renegotiation vanishes. This consumption path is different from the case of complete markets (i.e., full commitment), where consumption and debt are constant throughout. Interestingly, consumption is higher in the long run under limited commitment than under full commitment.

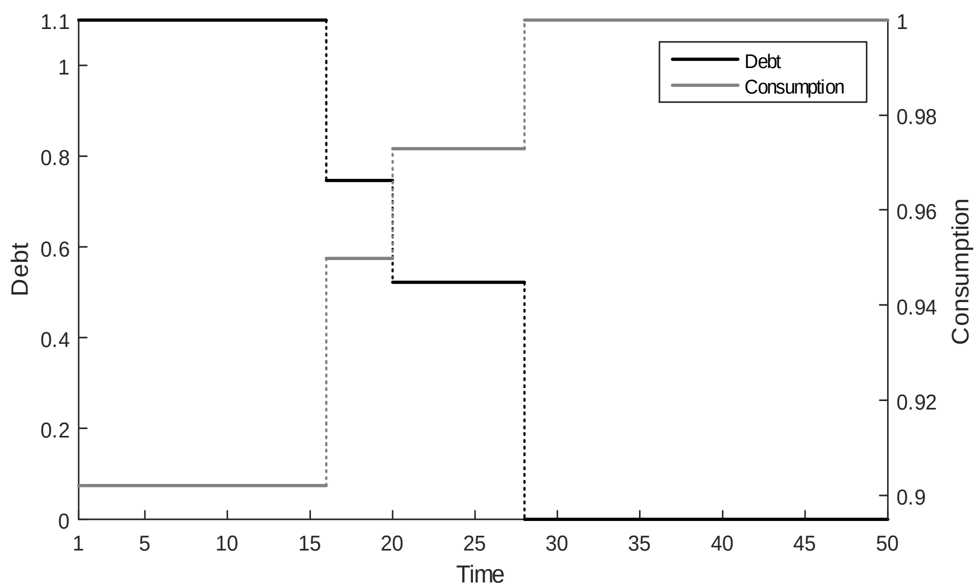

Figure 1: Simulation of debt and consumption for a particular sequence of ϕ's during normal times.

The prediction that whenever debt is renegotiated consumption increases permanently is extreme, and hinges on the assumption that ϕ is i.i.d. with a known distribution. In the online appendix (available upon request), we extend the model to a setting where there is uncertainty about the true distribution of ϕ and the market learns about this distribution by observing the sequence of ϕ's. In this case, a low realization of ϕ has two opposing effects on consumption: on the one hand, a low ϕ triggers debt renegotiation which on its own would increase consumption; on the other hand, a low ϕ affects the beliefs about the distribution of ϕ inducing the market to regard the country as less creditworthy (namely, the country draws from a distribution where low ϕ is more likely). This tends on its own to increase the default premium on bonds and to lower consumption.

The picture changes slightly if one assumes $\beta R < 1$. In this case, the economy accumulates debt even under full commitment. Thus, debt increases and consumption falls in periods in which debt is honored. After each round of renegotiation the economy is pushed back into the range of debt where the default risk is positive. In a world comprising economies with different β, e.g., some with $\beta R = 1$

and some with $\beta R < 1$, economies with low β (e.g., due to a shorter-sighted political process) would experience recurrent debt crises.

3.2 Equilibrium under recession

When the economy is in recession, the government chooses, sequentially, whether to honor the current debt, how much new debt to issue, and how much reform effort to exert. In this section, we assume that the government cannot issue state-contingent debt, i.e., securities whose payment is contingent on the aggregate state of the economy. In Section 5.1 below we relax this restriction.

3.2.1 Reform effort in equilibrium

The reform effort is chosen after new debt is issued, and is assumed to be non-contractible. The equilibrium price of debt incorporates the rational expectations of lenders about the reform effort. We denote by $\Psi(b')$ the equilibrium policy function for effort, or identically the probability that the recession ends in the next period, as a function of the newly-issued debt. More formally,

$$\Psi\left(b'\right) = \arg\max_{p} \left\{-X(p) + \beta\left[pEV\left(b', \bar{w}\right) + (1-p)E\left[V\left(b', \underline{w}\right)\right]\right]\right\}.$$

The first order condition yields:

$$X'\left(\Psi\left(b'\right)\right) = \beta\left[\int_0^\infty V\left(b', \phi', \bar{w}\right)dF(\phi) - \int_0^\infty V\left(b', \phi', \underline{w}\right)dF(\phi)\right]. \tag{12}$$

Characterizing the policy function Ψ requires that we solve for the value function V, which in turn depends on the equilibrium debt price. To this end, we solve first for the threshold realization of the renegotiation cost $\underline{\Phi}(b)$ when the economy remains in recession. This threshold satisfies the condition $W^H(b, \underline{w}) = W^D(\underline{\Phi}(b), \underline{w})$, which can be solved as:[9]

$$\underline{\Phi}(b) = \ln(\underline{w}) - X(\Psi(0)) + [1-\Psi(0)] \times \beta W^H(0, \underline{w}) + \Psi(0)\frac{\beta}{1-\beta}\ln(\bar{w}) - W^H(b, \underline{w}). \tag{13}$$

In the appendix we prove that $\bar{\Phi}(b) \geq \underline{\Phi}(b)$, and, hence, $F\left(\bar{\Phi}(b)\right) \leq F\left(\underline{\Phi}(b)\right)$. Intuitively, given b, there is a subset of realizations of ϕ such that the country renegotiates its debt if the recession continues but honors it if the recession ends. This observation allows us to partition the state space into three regions:

- in the low range, $b < b^-$, the country honors the debt with a positive probability, irrespective of the aggregate state (the probability of renegotiation being higher if the recession continues than if it ends);[10]
- in the intermediate range $b \in \left[b^-, \bar{b}\right)$, the country renegotiates with probability one if the recession continues, while it honors the debt with a positive probability if the recession ends;

[9] The analogous expression when the economy is in normal times is $\Phi(b, \bar{w}) = \ln(\bar{w}) + \frac{\beta}{1-\beta}\ln(\bar{w}) - W^H(b, \bar{w})$.

[10] b^- is implicitly determined by the equation $W^H\left(b^-, \underline{w}\right) = \inf_\phi W^D(\phi, \underline{w})$.

- in the high range $b \geq \bar{b}$, the country renegotiates its debt with probability one, irrespective of the aggregate state.

Note that limited commitment introduces some elements of state-contingencies, since debt is repaid with different probabilities under recession and normal times. This property is particularly stark when debt is in the intermediate range, where debt is renegotiated with probability one if the recession continues.

Consider, next, the bond price:

$$Q(b, \underline{w}) = \Psi(b) \times Q(b, \bar{w}) + [1 - \Psi(b)] \times \hat{Q}(b, \underline{w}) < Q(b, \bar{w}), \tag{14}$$

where $Q(b, \bar{w})$ is given by equation (5), and

$$\hat{Q}(b, \underline{w}) \equiv \frac{1}{R}(1 - F(\underline{\Phi}(b))) + \frac{1}{R}\frac{1}{b}\int_0^{\Phi(b)} \left(\underline{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right)$$

is the bond price conditional on the recession not ending, before the realization of ϕ is known. Note that $Q(b, \bar{w}) \geq \hat{Q}(b, \underline{w})$: if the recession ends, bonds become more valuable, because the probability of renegotiation is lower. This observation implies that the government underprovides reform effort relative to the case of contractible effort, because the creditors reap part of the gain from economic recovery, whereas the country bears the full burden of the effort cost. This can be established more formally with the aid of a simple one-period deviation argument. Consider an equilibrium effort choice path consistent with (12) – corresponding to the case of non-contractible effort. Next, suppose that, only in the initial period, the country can contract effort before issuing new debt. As it turns out, the country would choose a higher reform effort in the first period than in the equilibrium with non-contractible effort. We state this result as a lemma.

Lemma 3 *If $b' > 0$ and the borrower can, in the initial period, commit to an effort level upon issuing new debt, then the reform effort is strictly larger than in the case in which effort is never contractible.*

If the government could commit to reform, its reform effort would be monotone increasing in the debt level, since a high debt increases the hardship of a recession.[11] However, under moral hazard, the equilibrium reform effort exhibits a non-monotonic behavior. More precisely, $\Psi(b)$ is increasing at low levels of debt, and decreasing in a range of high debt levels, including the entire region $\left[b^{-}, \bar{b}\right]$. Proposition 2 establishes this result more formally.

Proposition 2 *There exist three ranges, $[0, b_1] \subseteq [0, b^{-}]$, $[b_2, \bar{b}] \supseteq [b^{-}, \bar{b}]$, and $\left[\bar{b}, \infty\right)$ such that:*

1. *If $b \in [0, b_1)$, $\Psi'(b) > 0$;*
2. *If $b \in \left(b_2, \bar{b}\right)$, $\Psi'(b) < 0$;*
3. *If $b \in \left[\bar{b}, \infty\right)$, $\Psi'(b) = 0$.*

The following argument establishes the result. Consider a low debt range where the probability of renegotiation is zero. In this range, there is no moral hazard.[12] Thus, a higher debt level has a

[11] In Section 4.1 below, we show that the reform effort would also be monotone increasing in the first best.

[12] If zero were in the support of the distribution of ϕ, the probability of renegotiation would be positive for all positive debt levels. However, a limit argument along the same lines applies as $b \to 0$.

disciplining effect, i.e., it strengthens the incentive for economic reforms: due to the concavity of the utility function, the discounted gain of leaving the recession is an increasing function of debt. As one moves to a larger initial debt, however, moral hazard becomes more prominent, since the reform effort decreases the probability of default, shifting some of the gains to the creditors. This is reminiscent of the *debt overhang* effect in Krugman (1988).

The debt overhang dominates over the disciplining effect in the region $[b^{-}, \bar{b}]$. In this range, if the economy remains in recession, debt is renegotiated for sure, and the continuation utility is independent of b. In contrast, if the recession ends, the continuation utility is decreasing in b. Thus, the value of reform effort necessarily declines in b. By continuity, the same argument extends to a range of debt below b^{-}. Finally, when $b > \bar{b}$, the economy renegotiates with probability one, and the gain from leaving the recession is independent of b. In a variety of numerical simulations, we have always found Ψ to be hump-shaped with a unique peak (see Figure 3), although we could not prove that hump-shapedness is a general property of the economy.

3.2.2 Debt issuance and consumption dynamics

Consider, next, consumption and the issuance of new debt. We start by establishing that the top of the Laffer curve of debt is lower in recession than during normal times.

Lemma 4 *Let $\bar{b} = \arg\max_b \{Q(b, \bar{w})\, b\}$ and $\bar{b}^R = \arg\max_b \{Q(b, \underline{w})\, b\}$ denote the top of the Laffer curve during normal times and recession, respectively. Then, $\bar{b}^R \leq \bar{b}$. In particular, if the probability of staying in a recession is exogenous (i.e., $\Psi(b) = p$), then $\bar{b}^R = \bar{b}$, otherwise, $\bar{b}^R < \bar{b}$.*

The reason why the top of the Laffer curve under recession is located strictly to the left of $\bar{b}$ when effort is endogenous is that the reform effort is decreasing in debt (i.e., $\Psi' < 0$) for b close to $\bar{b}$. Therefore, by reducing the newly-issued debt, the borrower can increase the subsequent reform effort, which in turn increases the current bond price and debt revenue.

Next, we discuss the CEE. We proceed in two steps, providing first an intuitive discussion of its properties, and then summarizing results in a formal proposition. The sovereign government solves the following problem:

$$
\begin{aligned}
B\left(\tilde{b}, \underline{w}\right) \quad = \quad & \arg\max_{b'} \left\{ \ln\left(Q\left(b', \underline{w}\right) \times b' + \underline{w} - \tilde{b}\right) - X\left(\Psi\left(b'\right)\right) \right. \\
& \left. + \beta\Psi\left(b'\right) \times EV\left(b', \bar{w}\right) + \beta\left(1 - \Psi\left(b'\right)\right) \times EV\left(b', \underline{w}\right) \right\},
\end{aligned} \tag{15}
$$

where $\tilde{b} = \min\left\{b, \hat{b}(\phi, \underline{w})\right\}$. Using the first order condition together with the envelope condition, and continuing to assume $\beta R = 1$, yields the following CEE:

$$
\begin{aligned}
& E\left\{\frac{MU_{t+1}}{MU_t} | \text{debt is honored at } t+1\right\} \\
= \quad & 1 + \frac{\Psi'(b_{t+1})}{\Pr(\text{debt is honored at } t+1)} R \left[Q(b_{t+1}, \bar{w}) - \hat{Q}(b_{t+1}, \underline{w})\right] b_{t+1}.
\end{aligned} \tag{16}
$$

Equation (16) is the analogue of (9). There are two differences. First, the expected ratio between the marginal utilities replaces the plain ratio between the marginal utilities, due to the uncertainty about the future aggregate state (recession or normal times). Second, there is a new term on the right-hand side capturing the effect of debt on reform effort.

For expositional purposes, it is useful to highlight first the properties of the case in which the probability that the recession ends is exogenous, so $\Psi'(b_{t+1}) = 0$. In this case, the CEE requires that the expected marginal utility be constant. For this to be true, consumption growth must be positive if the recession ends, and negative if the recession continues, namely,

$$\frac{c'|_{H,\bar{w}}}{c} > 1 > \frac{c'|_{H,\underline{w}}}{c}.$$

The lack of consumption insurance stems from the incompleteness of financial markets, and would disappear if the government could issue state-contingent bonds. However, this conclusion does not carry over to the economy with moral hazard, as we discuss in more detail in Section 5.1 below.

Consider, next, the general case. Moral hazard introduces a strategic motive in debt policy. By changing the level of newly-issued debt, the government strategically manipulates its own *ex-post* incentive to make reforms. The sign of this strategic effect is ambiguous, and hinges on the sign of Ψ' (see Proposition 2). When the outstanding debt is low, $\Psi' > 0$, more debt strengthens the *ex-post* incentive to reform, thereby increasing the price of the newly-issued debt. The right-hand side of (16) is in this case larger than unity, and the CEE implies a lower consumption fall (hence, higher debt accumulation) than in the absence of moral hazard. In contrast, in the region of high initial debt, $\Psi' < 0$, there is lower debt accumulation than in the absence of moral hazard. The reason is that the market anticipates that a larger debt reduces the reform effort. In response, the government restrains its debt accumulation strategically in order to mitigate the ensuing fall in the debt price. Thus, when the recession continues, a highly indebted country will suffer a deeper fall in consumption when the reform is endogenous than when the probability that the recession ends is exogenous.

We summarize the results in a formal proposition.[13]

Proposition 3 *If the economy starts in a recession, the following CEE holds true:*

$$\beta R \left(\begin{array}{c} \dfrac{\overbrace{\left[1-\Psi\left(b'\right)\right]\times\left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right)}^{}_{\text{prob. of repayment and continuing recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\text{unconditional prob. of repayment}}} \times \underbrace{\dfrac{c}{c'|_{H,\underline{w}}}}_{\text{marg. util. ratio if rec.}} \\ + \dfrac{\underbrace{\Psi\left(b'\right)\times\left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right)}_{\text{prob. of repayment and end of recession}}}{\underbrace{\Pr\left(H|b'\right)}_{\text{unconditional prob. of repayment}}} \times \underbrace{\dfrac{c}{c'|_{H,\bar{w}}}}_{\text{marg. util. ratio if n.t.}} \end{array} \right) = 1 + \frac{\Psi'\left(b'\right)}{\Pr\left(H|b'\right)} \times \underbrace{R\left(Q\left(b',\bar{w}\right)-\hat{Q}\left(b',\underline{w}\right)\right)b'}_{\text{gain to lenders if the economy recovers}} \tag{17}$$

where $c = C\left(b', \underline{w}; b\right)$, $c' = C^H\left(b', \underline{w}\right)$ *as defined in equations (10)-(11), and* $\Pr\left(H|b'\right)$ *is the unconditional probability that the debt* b' *be honored,*

$$\Pr\left(H|b'\right) = \left[1-\Psi\left(b'\right)\right]\times\left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right) + \Psi\left(b'\right)\times\left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right).$$

[13] Following Definition 1, the CEE here describes the expected ratio of the marginal utility of consumption in all states of nature such that ϕ' induces the government to honor its debt. Note that this set of realizations depends on the aggregate state.

3.2.3 Taking stock

This section has established the main properties of the competitive equilibrium. The first is that moral hazard induces underprovision of effort in equilibrium. The problem becomes more severe the larger the stock of debt is. Figure 2 shows the hump-shaped effort function, $\Psi(b)$, in a calibrated economy. The second is that the possibility of renegotiating a non-state-contingent debt may improve risk sharing, especially when debt is large. In particular, in the high-debt range $[b^-, \bar{b}]$, issuing renegotiable non-state-contingent debt goes in the direction of issuing different amounts of state-contingent debt paying a higher return if the recession ends than if it continues. Risk sharing would *per se* be welfare-enhancing. However, it exacerbates the moral hazard in reform effort.

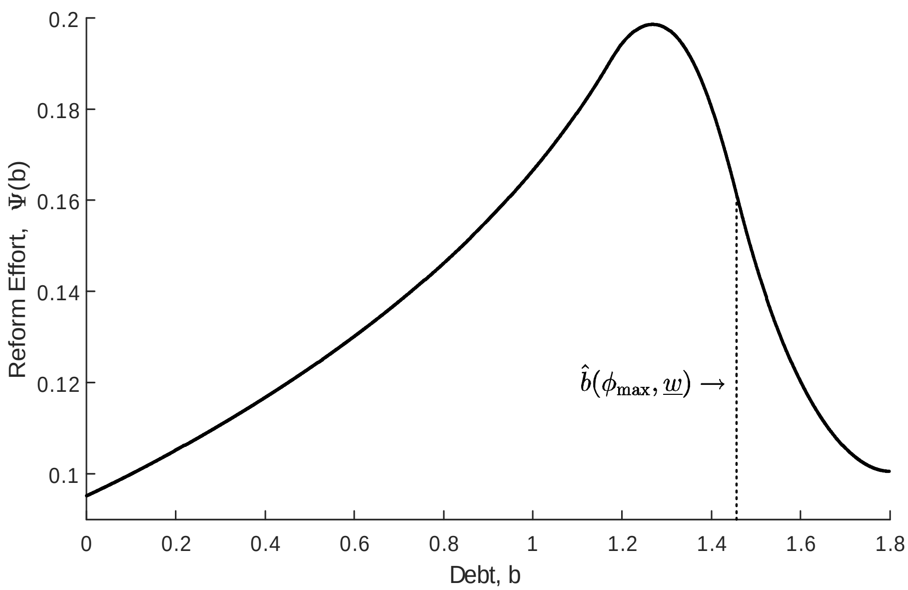

Figure 2: Reform effort function $\Psi(b)$. The parameter values correspond to those of the calibrated economy of Section 6.

The third property is that in periods in which debt is fully honored, the equilibrium features positive debt accumulation if the economy remains in recession, and constant debt when the economy returns to normal times. An implication of the first and third property is that, as the recession persists, the reform effort initially increases, but then, for high debt levels, it declines over time. Figure 3 shows the time path of debt and consumption (left panel) and of the corresponding reform effort (right panel) for a particular sequence of ϕ's. The recession ends at time T.

The fourth property concerns post-renegotiation dynamics. Consumption may increase after a sufficiently large haircut, even though the recession does not come to an end. However, in this case debt accumulation resumes right after the haircut. This prediction is consistent with the debt dynamics of Greece after the 2011 haircut discussed in the introduction. Interestingly, a large haircut may in some cases increase the reform effort, contrary to the common view that pardoning debt always has perverse effects on incentives.

Although, for tractability, we assume that the recession is totally unanticipated, it is interesting to compare two economies entering a surprise recession with different debt levels. Initially, both

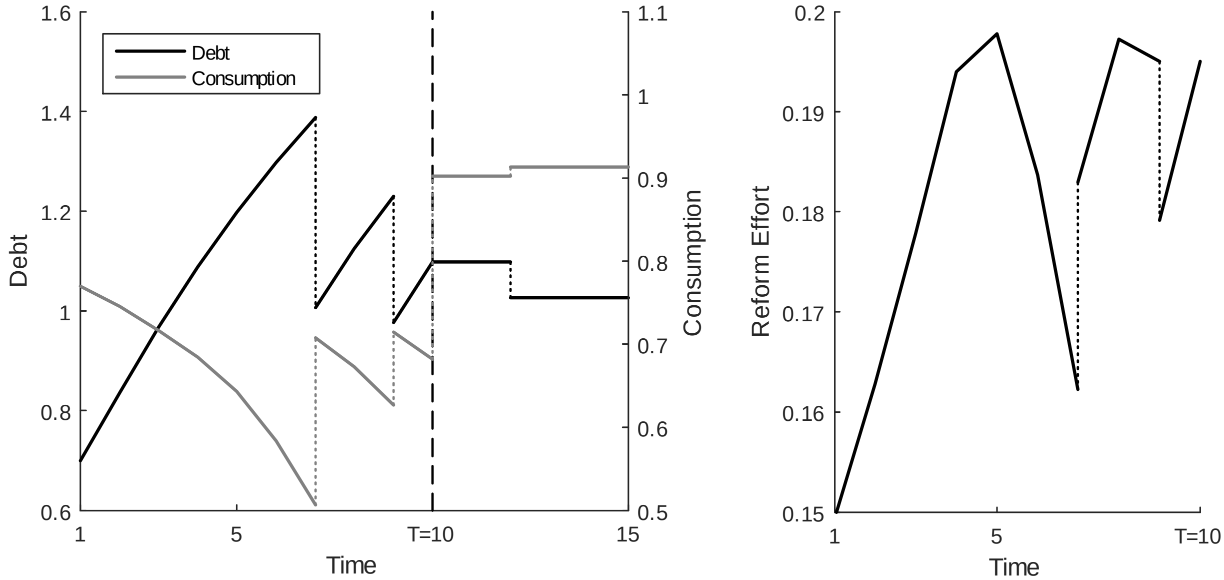

Figure 3: Simulation of debt, consumption and effort for a particular seuence of ϕ's in the competitive equilibrium. Here, the recession ends at time $T = 10$.

economies experience a falling consumption and a growing debt. However, the low-debt country may stay (at least temporarily) in the region where debt is repaid with probability one. Then, in the high-debt country, the effect of the recession is aggravated by a soaring interest rate, while this does not happen in the low-debt country. Consequently, unless there is renegotiation, consumption falls much faster in the country with a high initial debt. This is consistent with the observation that the European debt crisis has hit especially hard consumption in countries which entered the recession with an already high debt.

4 Efficiency

In this section, we study the efficient allocation and compare it with the competitive equilibrium. We start by characterizing the first-best allocation. Then, we characterize the constrained efficient allocation in an environment where the planner cannot overrule the limited commitment constraint. This is a useful benchmark, since in reality international agencies (e.g., the IMF) can observe and possibly monitor countries' reforms but have limited instruments to prevent sovereign debt renegotiation.

4.1 First Best

The first best entails perfect insurance: the country enjoys a constant stream of consumption and exerts a constant reform effort during recession. For comparison with the constrained efficient allocation studied below, it is useful to write the problem in terms of a dynamic principle-agent framework. To this aim, let ν^{FB} denote the discounted utility that the planner is committed to deliver to the country (i.e., the "promised utility") and let p^{FB} denote the probability that the recession ends. The superscript FB refers to "first best." Then:

$$\nu^{FB} = \frac{u\left(c^{FB}\right)}{1-\beta} - \frac{1}{1-\beta\left(1-p^{FB}\right)} X\left(p^{FB}\right). \tag{18}$$

The planner maximizes the principal's profit,

$$\underline{P}^{FB} = \underline{w} - c + \beta\left(1-p\right)\underline{P}^{FB} + \beta p \bar{P}^{FB}, \tag{19}$$

subject to the promise-keeping constraint that $\nu \geq \nu^{FB}$. Here, $\bar{P}^{FB}(\nu)$ and $\underline{P}^{FB}(\nu)$ denote the expected present value of profits accruing to the (risk-neutral) principal in normal times and recession, respectively, conditional on delivering the promised utility ν in the most efficient way. In normal times, $\bar{P}^{FB} = \left(\bar{w} - c^{FB}\right)/\left(1-\beta\right)$. Writing the Lagrangian and applying standard methods yields the following lemma.

Lemma 5 *Consider an economy starting in recession. The optimal contract (first best) satisfies the following trade-off between consumption and reform effort:*

$$\frac{\beta}{1-\beta\left(1-p^{FB}\right)}\left(\underbrace{\left(\bar{w}-\underline{w}\right)\times u'\left(c^{FB}\right)}_{\textit{increase in profits if econ. recovers}} + \underbrace{X\left(p^{FB}\right)}_{\textit{saved effort cost if econ. recovers}}\right) = X'\left(p^{FB}\right). \tag{20}$$

$X'\left(p^{FB}\right)$ is the marginal cost of increasing the probability of recovery. The marginal benefit (left-hand side) comprises two terms. The first term is the discounted value of the extra profit accruing to the principal if the recession ends, expressed in units of consumers' utils. The second term is the discounted gain accruing to the agent from dispensing with the reform effort. Perfect insurance implies that no consumption gain accrues to consumers when the recession ends.

Combining (20) and (18) yields the complete characterization of the first best. After rearranging terms, one obtains:

$$X'\left(p^{FB}\right)\left(\frac{1-\beta}{\beta} + p^{FB}\right) - X\left(p^{FB}\right) = \left(\bar{w}-\underline{w}\right)\times u'\left(c^{FB}\right) \tag{21}$$

$$\frac{1}{1-\beta\left(1-p^{FB}\right)} X\left(p^{FB}\right) = \frac{u\left(c^{FB}\right)}{1-\beta} - \nu^{FB} \tag{22}$$

Equation (21) defines a negatively sloped locus in the plane $\left(p^{FB}, c^{FB}\right)$, while equation (22) defines a positively sloped locus in the same plane. Under appropriate conditions, the two equations pin down a unique interior solution for p and c (otherwise, the optimal effort is zero). The comparative statics with respect to ν^{FB} is especially interesting. An increase in ν^{FB} yields an increase in c^{FB} and a reduction in p^{FB}, i.e., more consumption and less effort.

Note that ν^{FB} can be mapped into an initial debt level: a highly indebted country has a low ν^{FB} and, hence, a low consumption and a high reform effort. This finding contrasts with the competitive equilibrium where the relationship between debt and reform effort is hump-shaped.

4.2 Constrained Pareto optimum

In this section, we characterize the optimal dynamic contract, subject to limited commitment: the country can quit the contract, suffer the default cost, and resort to market financing. The problem is formulated as a one-sided commitment with lack of enforcement, following Ljungqvist and Sargent (2012) and based on a promised-utility approach in the vein of Spear and Srivastava (1987), Thomas and Worrall (1988) and Kocherlakota (1996).[14]

Here, ν denotes the promised utility to the risk-averse agent in the beginning of the period, before the realization of ϕ. ν is the key state variable of the problem. We denote by $\bar{\omega}_{\phi}$ and $\underline{\omega}_{\phi}$ the promised continuation utilities conditional on the realization ϕ and on the aggregate state $\bar{w}$ and $\underline{w}$, respectively.[15] $\underline{P}(\nu)$ and $\bar{P}(\nu)$ denote the expected present value of profits accruing to the principal conditional on delivering the promised utility ν in the most cost-effective way in recession and in normal times, respectively. The planning problem is evaluated after the uncertainty about the aggregate state has been resolved (i.e., the economy is either in recession or in normal times in the current period), but before the realization of ϕ is known. We continue to focus on the case in which $\beta R = 1$.

4.2.1 Constrained efficiency in normal times

In normal times, the optimal value $\bar{P}(v)$ satisfies the following functional equation:

$$\bar{P}(v) = \max_{\left\{\omega_{\phi}, \bar{w}, c_{\phi}\right\}_{\phi \in \aleph}} \int_{\aleph} \left[\bar{w} - c_{\phi} + \beta \bar{P}(\bar{\omega}_{\phi})\right] dF(\phi), \tag{23}$$

where the maximization is subject to the constraints

$$\int_{\aleph} \left[u(c_{\phi}) + \beta \bar{\omega}_{\phi}\right] dF(\phi) \quad \geq \quad v, \tag{24}$$

$$u(c_{\phi}) + \beta \bar{\omega}_{\phi} \quad \geq \quad \bar{v} - \phi, \tag{25}$$

where $\bar{v}$ is the value of "autarky" for the agent ($\bar{\nu} = W^{H}(0, \bar{w})$). The former is a promise-keeping constraint, whereas the latter is a participation constraint (PC). In addition, the problem must satisfy the constraints that $0 \leq c_{\phi} \leq \bar{w}$ and $\bar{\omega}_{\phi} \leq \bar{v}$. The problem has standard properties: the constraint set is convex, while the one-period return function in (23) is concave. In the online appendix, we prove that the profit function $\bar{P}(v)$ (and its analogue under recession, $\underline{P}(v)$) is decreasing, strictly concave and twice differentiable. The application of recursive methods allows us to establish the following proposition.

Proposition 4 *Assume the economy is in normal times. (I) For all states s such that the PC of the agent, (25), is binding, $\bar{\omega}_{\phi} > \nu$ and the solution for $(c_{\phi}, \bar{\omega}_{\phi})$ is determined by the following conditions:*

$$u'(c_{\phi}) = -\frac{1}{\bar{P}'(\bar{\omega}_{\phi})}, \tag{26}$$

$$u(c_{\phi}) + \beta \bar{\omega}_{\phi} = \bar{v} - \phi. \tag{27}$$

The solution is not history-dependent, i.e., the initial promise, v, does not matter. (II) For all realizations ϕ such that the PC of the agent, (25), is binding, $\bar{\omega}_{\phi} = \nu$ and $c_{\phi} = c(\nu)$. The solution is history-dependent.

[14] Adding a participation constraint for the planner would not affect the solution, since such constraint is never binding. Thus, the problem can as well be interpreted as a two-sided limited commitment program.

[15] The states of nature in this problem correspond to the realizations of ϕ in the decentralized equilibrium.

The efficient allocation has standard properties. Whenever the agent's PC is not binding, consumption and promised utility remain constant over time. Whenever the PC binds, the planner increases the agent's consumption and promised utility in order to meet her PC.

In normal times, the constrained efficient allocation of Proposition 4 is identical to the competitive equilibrium. To establish this result, we return, first, to the competitive equilibrium. Let

$$\bar{\Pi}(b) = \left(1 - F\left(\bar{\Phi}(b)\right)\right) b + \int_0^{\bar{\Phi}(b)} \hat{b}(\phi, \bar{w})\, dF(\phi) \tag{28}$$

denote the expected value for the creditors of an outstanding debt b before the current-period uncertainty is resolved. Note that $\bar{\Pi}(b)$ yields the expected debt repayment, which is lower than the face value of debt, since in some states of nature debt is renegotiated. Recall that $EV(b, \bar{w}) = \int_{\aleph} V(b, \phi, \bar{w})\, dF(\phi)$ denotes the discounted utility accruing to a country with the debt level b in the competitive equilibrium. To prove the equivalence, we postulate that $\bar{\Pi}(b) = \bar{P}(\nu)$, and show that in this case $v = EV(b, \bar{w})$. If the equilibrium were not constrained efficient, the planner could do better, and we would find that $v > EV(b, \bar{w})$.

Proposition 5 *Assume that the economy is in normal times. The competitive equilibrium is constrained Pareto efficient, namely,* $\bar{\Pi}(b) = \bar{P}(\nu) \Leftrightarrow v = EV(b, \bar{w})$.

Intuitively, renegotiation provides the market economy with sufficiently many state contingencies to attain second-best efficiency. This result hinges on two features of the renegotiation protocol. First, renegotiation averts any real loss associated with unordered default. Second, creditors have all the bargaining power in the renegotiation game.[16]

4.2.2 Constrained efficiency in recession

Next, we consider an economy in recession. The principal's profit obeys the following functional equation:

$$\underline{P}(\nu) = \max_{\left\{\bar{\omega}_\phi, \underline{\omega}_\phi, c_\phi, p_\phi\right\}_{\phi \in \aleph}} \int_{\aleph} \left[\underline{w} - c_\phi + \beta\left((1 - p_\phi)\, \underline{P}\left(\underline{\omega}_\phi\right) + p_\phi \bar{P}\left(\bar{\omega}_\phi\right)\right)\right] dF(\phi), \tag{29}$$

where the maximization is subject to the constraints

$$\int_{\aleph} \left(u(c_\phi) - X(p_\phi) + \beta\left((1 - p_\phi)\, \underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right)\right) dF(\phi) \quad \geq \quad v, \tag{30}$$

$$u(c_\phi) - X(p_s) + \beta\left((1 - p_\phi)\, \underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) \quad \geq \quad \underline{\nu} - \phi, \tag{31}$$

and where $\underline{\nu} = W^H(0, \underline{w})$ is the value for the agent of breaking the contract when the economy is in recession. Note that there are two separate promised utilities, $\underline{\omega}_\phi$ and $\bar{\omega}_\phi$, associated with the two possible realizations of the aggregate state in the next period. The following proposition can be established.

[16] Both assumptions may be violated in the real world. For instance, renegotiations may entail costs associated with legal proceeds and lawsuits, trade retaliation, temporary market exclusion, etc. These would affect the strong efficiency results in a fairly obvious way. Also, creditors may be unable to force the country to its reservation utility in the renegotiation stage. This may reduce the amount of loans creditors can recover, increasing the ex-ante risk premium. In this case, the competitive equilibrium would fail to implement the second best.

Proposition 6 *Assume the economy is in recession. (I) For all realizations ϕ such that the PC of the agent, (25), is binding, the optimal choice vector $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ satisfies the following conditions:*

$$\begin{aligned}
u'\left(c_\phi\right) &= -\frac{1}{\underline{P}'\left(\underline{\omega}_\phi\right)}, &(32)\\
u\left(c_\phi\right) - X\left(p_\phi\right) + \beta\left(\left(1-p_\phi\right)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) &= \underline{\nu} - \phi, &(33)\\
\underline{P}'\left(\underline{\omega}_\phi\right) &= \underline{P}'\left(\bar{\omega}_\phi\right), &(34)\\
X'\left(p_\phi\right) &= \beta\left(u'\left(c_\phi\right)\left(\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right). &(35)
\end{aligned}$$

The solution is not history-dependent, i.e., the promised utility ν does not affect the solution. (II) For all realizations ϕ such that the agent's PC, (25), is not binding, $\underline{\omega}_\phi = \nu$, $\bar{\omega}_\phi = \bar{\omega}(\nu)$, $c_\phi = \underline{c}(\nu)$ and $p_\phi = p(\nu)$. The solution is history-dependent. The reform effort is decreasing in the promised utility level. (III) For all $\phi \in \aleph$, $\bar{\omega}_\phi > \underline{\omega}_\phi$.

When the agent's PC is slack, consumption, reform effort, and promised utilities remain constant over time. Every time the PC binds, the planner increases the promised utilities, and grants the agent an increase in consumption and a reduction in the reform effort. Relative to the first best, the agent is offered lower consumption and required to exercise higher effort as she enters the contract. The conditions faced by the agent are improved over time thereafter. Note that, if we compare two countries entering the contract with different initial promised utilities, the country with a lower promised utility earns a lower consumption and is asked to exercise higher effort. Thus, the country with the lower promised utility (i.e., a higher initial debt) is expected to recover faster from the recession.

Consider, next, the period in which the recession ends (part III of Proposition 6). As the recession ends, the promised utility increases and effort goes to zero. Consumption may either remain constant or increase depending on whether the PC binds. Interestingly, the set of states such that the PC binds expands. Namely, there are realizations of ϕ such that consumption rises and effort falls only if the recession ends. In contrast, for sufficiently large ϕ's, the agent's PC is binding irrespective of whether the recession continues or ends. In this case, consumption remains constant. In other words, because of limited commitment, the agent is offered some partial, but not perfect insurance against the continuation of the recession.

4.2.3 Comparison between constrained optimum and competitive equilibrium

The competitive equilibrium is not constrained Pareto efficient. In the competitive equilibrium, consumption falls over time during recession even when the country honors its debt. In contrast, the planner would insure the agent's consumption by keeping it constant. Therefore, the market under-provides insurance. The dynamics of the reform effort are also sharply different. In the constrained efficient allocation, effort is a monotone decreasing function of promised utility. Since promised utility is an increasing step function over time, effort is step-wise decreasing. In contrast, in the competitive equilibrium the reform effort is hump-shaped in debt. Since debt increases over time (unless it is renegotiated), effort is also hump-shaped over time conditional on no renegotiation.

Figure 4 displays the time path of consumption and effort (left panel) and of the corresponding promised utilities (right panel) for a particular sequence of ϕ's in the constrained efficient allocation. The dynamics are in sharp contrast with those of the competitive equilibrium in Figure 3.

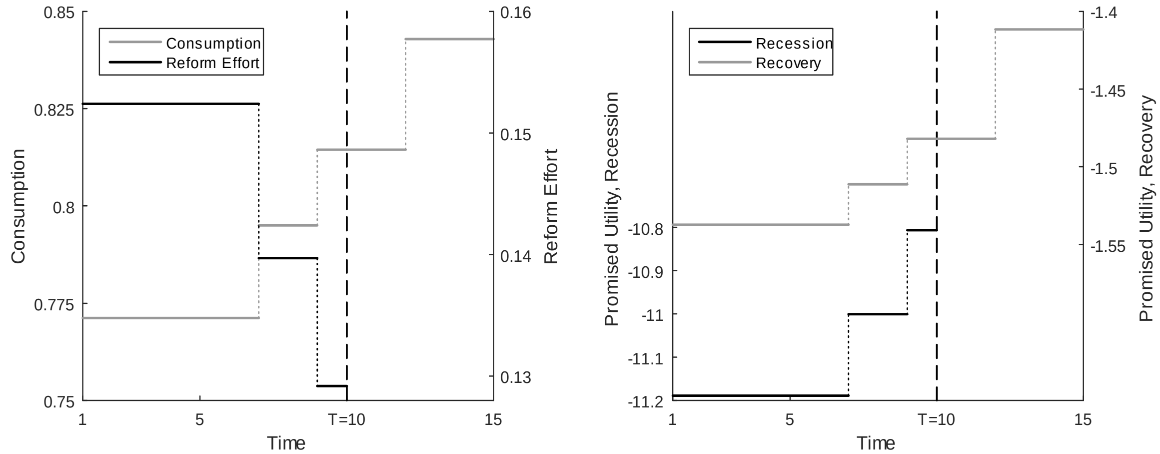

Figure 4: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's in the constrained optimum. Here, the recession ends at time $T = 10$.

5 Decentralization

In this section, we discuss policies and institutions that decentralize the constrained efficient allocation.

5.1 Laissez-faire equilibrium with state-contingent debt

The analysis of the laissez-faire equilibrium in Section 3 was carried out under the assumption that the government can only issue one non-contingent asset. In this section, we show that a laissez-faire equilibrium with state-contingent debt would attain constrained efficiency if and only if there were no moral hazard.[17] However, when the reform effort is endogenous, the combination of moral hazard and limited commitment curtails the insurance that markets can provide. Consequently, the laissez-faire equilibrium with state-contingent debt is not constrained efficient. In the quantitative analysis of Section 6 below, we show that markets for state-contingent debt yield only small quantitative welfare gains relative to the benchmark economy.

Let $b_{\underline{w}}$ and $b_{\bar{w}}$ denote Arrow securities paying one unit of output if the economy is in a recession or in normal times, respectively. We label these securities *recession-contingent debt* and *recovery-contingent debt*, respectively, and denote by $Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ and $Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ their corresponding prices. The budget constraint in a recession is given by:

$$Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} = b_{\underline{w}} + c - \underline{w}.$$

To establish a benchmark, consider first a complete market environment in which there is neither moral hazard nor limited commitment. In this case, the security $b'_{\underline{w}}$ sells at the price $Q_{\underline{w}} = (1-p)/R$ whereas the security $b'_{\bar{w}}$ sells at the price $Q_{\bar{w}} = p/R$. In equilibrium, consumption is constant over

[17]Even in the absence of moral hazard, state-contingent debt does not yield full insurance, due to the risk of renegotiation. However, it attains the second best.

time and across states. The equilibrium attains the first best.[18]

Under limited commitment, the price of each security depends on both outstanding debt levels, as both affect the reform effort and the probability of renegotiation.[19] The value function of the benevolent government can be written as:

$$\begin{aligned} V\left(b_{\underline{w}},\phi,w\right) &= \max_{\left\{b'_{\underline{w}},b'_{\bar{w}}\right\}} \left\{\ln\left[Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\bar{w}} + \underline{w} - \min\left\{b_{\underline{w}},\hat{b}\left(\phi,\underline{w}\right)\right\}\right]\right. \\ &\quad \left. -X\left(\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right) + \beta\left[1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right]EV\left(b'_{\underline{w}},\underline{w}\right) + \beta\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)EV\left(b'_{\bar{w}},\bar{w}\right)\right\}. \end{aligned} \quad (36)$$

Mirroring the analysis in the case of non-state-contingent debt, we proceed in two steps. First, we characterize the optimal reform effort. This is determined by the difference between the discounted utility conditional on the recession ending and continuing, respectively (cf. Equation (12)):

$$X'\left(\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right) = \beta\left[\int_0^\infty V\left(b'_{\bar{w}},\phi',\bar{w}\right)dF(\phi) - \int_0^\infty V\left(b'_{\underline{w}},\phi',\underline{w}\right)\, dF(\phi)\right].$$

Note that the incentive to reform would vanish under full insurance.

Next, we characterize the consumption and debt policy. To this aim, consider first the equilibrium asset prices. The prices of the recession- and recovery-contingent debt are given by, respectively:

$$Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right) = \frac{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) + \frac{1}{b'_{\underline{w}}}\int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)}\left(\underline{\Phi}^{-1}(\phi)\, dF(\phi)\right)\right), \quad (37)$$

$$Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right) = \frac{\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) + \frac{1}{b'_{\bar{w}}}\int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)}\left(\bar{\Phi}^{-1}(\phi)\, dF(\phi)\right)\right). \quad (38)$$

Operating like in Section 3, we determine the consumption and debt dynamics conditional on the continuation and on the end of the recession. The next proposition characterizes the CEE with state-contingent debt.

Proposition 7 *Assume $\beta R = 1$, and assume that there exist markets for two Arrow securities delivering one unit of output if the economy is in recession and in normal times, respectively, and subject to the risk of renegotiation. Suppose that the economy is initially in recession. The following CEEs are satisfied in the competitive equilibrium: (I) If the recession continues and debt is honored next period, consumption growth is given by:*

$$\underbrace{\frac{c}{c'|_{H,\underline{w}}}}_{\text{marg. util. ratio if rec.}} = 1 + \underbrace{\Psi_{b'_{\underline{w}}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)}_{>0}\times\underbrace{\frac{R\times\Delta\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left(1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right)}}_{>0}. \quad (39)$$

[18] Given an outstanding recession-contingent debt $b_{\underline{w}}$, the equilibrium features:

$$\begin{aligned} b'_{\underline{w}} &= b_{\underline{w}} \\ b'_{\bar{w}} &= b_{\underline{w}} + \frac{R}{R-(1-p)}\left(\bar{w}-\underline{w}\right) \\ c &= \underline{w} + \frac{p}{R-(1-p)}\left(\bar{w}-\underline{w}\right) - \frac{R-1}{R}b_{\underline{w}} \end{aligned}$$

[19] Note that these assets are not Arrow-Debreu assets since their payoffs are conditional on the realization of ϕ. An alternative approach would have been to follow Alvarez and Jermann (2000) and issue an Arrow-Debreu asset for each state (w,ϕ) and let the default-driven participation constraint serve as an endogenous borrowing constraint.

(II) If the recession ends and debt is honored next period, consumption growth is given by:

$$\underbrace{\frac{c}{c'|_{H,\bar{w}}}}_{marg.\ util.\ ratio\ if\ n.t.} = 1 + \underbrace{\Psi_{b'_{\bar{w}}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}_{<0} \times \underbrace{\frac{R \times \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}}_{>0}. \tag{40}$$

The term $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right)$ *is given by*

$$\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \equiv \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} - \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} \geq 0. \tag{41}$$

Moreover,

$$\begin{aligned}
c &= Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \tilde{b}, \\
c'|_{H,\underline{w}} &= Q_{\underline{w}}\left(B_{\bar{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\underline{w}}\left(b'_{\underline{w}}\right) + Q_{\bar{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\bar{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b'_{\underline{w}}, \\
c'|_{H,\bar{w}} &= Q\left(B\left(b'_{\bar{w}}, \bar{w}\right), \bar{w}\right) \times B\left(b'_{\bar{w}}, \bar{w}\right) + \bar{w} - b'_{\bar{w}},
\end{aligned}$$

$B_{\underline{w}}\left(b_{\underline{w}}\right)$ *and* $B_{\bar{w}}\left(b_{\underline{w}}\right)$ *denote the optimal level of newly-issued recession- and recovery-contingent debt when the recession continues, debt is honored, and the outstanding debt level is* $b_{\underline{w}}$*.*

Without moral hazard (i.e., if the probability that the recession ends is exogenous, and $\Psi_{b'_{\underline{w}}} = \Psi_{b'_{\bar{w}}} = 0$), consumption would be independent of the realization of the aggregate state as long as the government honors its debt. In this case, the CEEs imply constant consumption $c'|_{H,\underline{w}} = c'|_{H,\bar{w}} = c$ when the debt is honored. The solution has the same properties as the constrained Pareto optimum without moral hazard: consumption is constant when debt is honored, and increases discretely when it is renegotiated. The next proposition establishes formally that the two allocations are equivalent. To this aim, define $\underline{\Pi}(b_w)$ to be the expected value of debt conditional on staying in recession but before the realization of ϕ.

Proposition 8 *If the probability that the recession ends is independent of the reform effort (i.e.,* $\Psi = p$*), then the competitive equilibrium with state-contingent debt is constrained Pareto efficient, namely,* $\underline{\Pi}(b_w) = \underline{P}(\nu) \Leftrightarrow v = EV(b_w, \underline{w})$*.*

This equivalence breaks down if there is moral hazard. In this case, consumption and effort dynamics are qualitatively different across the two allocations. In the competitive equilibrium, consumption falls (and recession-contingent debt increases) whenever the economy remains in recession and debt is honored. This follows from equation (39). By increasing the recession-contingent debt, the country strengthens its incentive to exert reform effort, since $\Psi_{b'_{\underline{w}}} > 0$. This induces the government to issue more recession-contingent debt. The effect is stronger the larger the term Δ is which yields the net expected gain accruing to the lenders from a marginal increase in the probability that the recession ends. As far as the recovery-contingent debt is concerned, Equation (40) implies, since $\Psi_{b'_{\bar{w}}} > 0$, that consumption grows if the recession ends. The reason is that a reduction in the newly-issued recovery-contingent debt strengthens the incentive to reform. This result highlights the trade-off between insurance and incentives: the country must give up insurance in order to gain credibility about its willingness to do reforms. It also implies that the allocation is not constrained efficient, since, recall, in the planner allocation consumption is constant when the outside option is not binding, and increases discretely when the latter is binding. In summary, the decentralized equilibrium is inefficient and provides less consumption smoothing than does the planner.[20]

[20] The behavior of effort is also different between the equilibrium and the constrained efficient allocation. In the planning

5.2 An austerity program

The market failure in the previous section arises from the moral hazard in reform effort. The constrained efficient allocation would be decentralized by the competitive equilibrium if, in addition, effort were contractible. In reality, it seems difficult that a country can issue state-contingent bonds in the market while committing credibly to future reforms. In this section, we discuss an institutional arrangement that implements the efficient allocation through the enforcement of an international institution that can monitor the reform effort, but not overrule the limited commitment problem.

Consider a stand-by program implemented by an international institution (e.g., the IMF). The indebted country can decide to quit the stand-by program unilaterally. We show that a combination of transfers (or loans), repayment schedule and renegotiation strategy can implement the constrained optimal allocation. This program has two key features. First, the country cannot run an independent fiscal policy, i.e., it is not allowed to issue additional debt in the market. Second, the program is subject to renegotiation. More precisely, whenever the country can credibly threaten to abandon the program, the international institution should sweeten the deal by increasing the transfers and reducing the required effort, and reducing the debt the country will be settled with when the recession ends. When no credible threat of default is on the table, consumption and reform effort should be held constant as long as the recession lasts. When the recession ends, the international institution receives a payment from the country.

Let ν denote the present discounted utility guaranteed to the country when the program is initially agreed upon. Let $c^*(\nu)$ and $p^*(\nu)$ be the consumption and reform effort associated with the promised utility in the planning problem. Upon entering the program, the country receives a transfer equal to $T(\nu) + b_0$, where $T(\nu) = c^*(\nu) - \underline{w}$ (note that $T(\nu)$ could be negative). In the subsequent periods, the country is guaranteed the transfer flow $T(\nu)$ so long as the recession lasts and there is no credible request of renegotiating the terms of the austerity program. In other words, the international institution first bails out the country from its obligations to creditors, and then becomes the sole residual claimant of the country's sovereign debt. The country is also asked to exercise a reform effort $p^*(\nu)$. If the country faces a low realization of ϕ and threatens to leave the program, the institution improves the terms of the program so as to match the country's outside option. Thereafter, consumption and effort are held constant at new higher and lower level, respectively, as in the planner's allocation. And so on, for as long as the recession continues.

As soon as the recession ends, the country owes a debt b_N to the international institution, determined by the equation

$$Q(b_N, \bar{w}) \times b_N = c^*(\nu_N) - \bar{w} + b_N.$$

Here ν_N is the expected utility granted to the country after the most recent round of renegotiation. After receiving this payment, the international institution terminates the program and lets the country finance its debt in the market.

This program resembles an *austerity* program, in the sense that the country is prevented from running an independent fiscal policy. In particular, the country would like to issue extra debt after entering the stand-by agreement, so austerity is a binding constraint. In addition, the country would like to shirk on the reform effort prescribed by the agreement. Thus, the government would like to deviate from the optimal plan, and an external enforcement power is an essential feature of the

allocation, effort is constant whenever the outside option is not binding. In contrast, in the decentralized allocation, changes in debt will generally influence the reform effort, which is increasing in the newly-issued recession-contingent debt, $\Psi_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$, and decreasing in the newly-issued the recovery-contingent debt, $\Psi_{b'_{\bar{w}}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) < 0$.

program. This conflict of interest rationalizes the tense relationship between the Greek government and the Troika since the stipulation of the stand-by agreement.

A distinctive feature of the assistance program is that the international institutions sets "harsh" entry conditions in anticipation of future renegotiations. How harsh such conditions are depends on ν. In turn, ν may reflect a political decision about how many (if any) own resources the international institution wishes to commit to rescuing the indebted country. A natural benchmark is to set ν such that the international institution makes zero profits (and zero losses) in expected discounted value. Whether, ex-post, the international institution makes net gains or losses hinges on the duration of the recession and on the realized sequence of ϕ's.

Another result that has important policy implications is that there would be no welfare gain if the international institution committed to never accepting any renegotiation. On the contrary, such a policy would lead to welfare losses. The reason is that, on the one hand, there would be inefficient default in equilibrium. On the other hand, the country could not expect future improvements, and would therefore not accept a very low initial consumption, or a very high reform effort. If one fixes the expected profit of the international institution to zero, the country would receive a lower expected utility from the alternative program.

In summary, our theory prescribes a pragmatic approach to debt renegotiation. Any credible threat of default should be appeased by reducing the debt and softening the austerity program. Such approach is often criticized for creating bad incentives. In our model, it is instead the optimal policy under the reasonable assumption that penalties on sovereign countries for breaking an agreement are limited.

5.3 Self-enforcing reform effort

Thus far we have assumed that the planner – or the international institution in the decentralized environment – can dictate the reform effort as long as the country stays within the contract. Assuming that reforms are observable seems natural to us. It is possible, for instance, to verify whether Greece introduces labor market reforms, cuts employment in the public sector, or passes legislative measures to curb tax evasion (e.g., by intensifying tax audits and enforcing penalties). Nevertheless, it may difficult for international institutions to prevent deviations such as delays, lack of implementation, or weak enforcement of reforms that were agreed upon. In other words, the borrower may try to cash-in the transfer agreed in the assistance program in exchange for promises of structural reforms, but indefinitely defer their execution.

In this section, we consider an alternative environment where reform effort can only be verified *ex-post* at the end of each period. If the planner detects shirking, she terminates the program irreversibly.[21] A new incentive-compatibility constraint (IC) arises from the inability of the country to commit to reforms. In particular, the country could behave opportunistically by cashing the loan at the beginning of the period and exercise a discretionary effort level. In this case, the government would be forced to revert to the market equilibrium with their debt obligations restored to the level prior to the start of the assistance program. The appeal of such a deviation is larger the higher requested reform effort is.

More formally, the allocation is identical to the solution to the planning problem (29)–(31) subject to the additional IC stipulating that, for all $\phi \in \aleph$,

$$-X(p_\phi) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right) \geq Z(b_0), \tag{42}$$

[21] One could consider less severe punishments, such as a temporary suspension of the loans, or a reduction in the future loans. We leave these extensions to future research.

where b_0 is the debt of the country when it enters the contract, and Z is the continuation utility if the economy reverts to the competitive equilibrium, i.e.,[22]

$$Z(b_0) \equiv -X(\Psi(b_0)) + \beta\left[\Psi(b_0) \times E\left[V\left(b_0, \phi', \bar{w}\right)\right] + (1-\Psi(b_0)) \times E\left[V\left(b_0, \phi', \underline{w}\right)\right]\right].$$

When the IC (42) is binding, the allocation of Proposition 6 is susceptible to profitable deviations.[23] The following Lemma establishes properties of the constrained allocation whenever the IC is binding.[24]

Lemma 6 *When the IC is binding, effort and promised utilities are constant at the levels $(p^*, \underline{\omega}^*, \bar{\omega}^*)$, where the triplet is uniquely determined by the equations*

$$\begin{aligned} Z(b_0) &= -X(p^*) + \beta\left((1-p^*)\underline{\omega}^* + p^*\bar{\omega}^*\right), && (43)\\ \bar{P}'(\bar{\omega}^*) &= \underline{P}'(\underline{\omega}^*), && (44)\\ X'(p^*) &= \beta(\bar{\omega}^* - \underline{\omega}^*) - \frac{\beta}{\underline{P}'(\underline{\omega}^*)}\left(\bar{P}(\bar{\omega}^*) - \underline{P}(\underline{\omega}^*)\right), && (45) \end{aligned}$$

where the profit functions $\bar{P}$ and $\underline{P}$ are defined as in Section 4.2.

Equation (43) yields the IC when it holds with equality. Equations (44) and (45) then follow from the FOCs stated in equations (82)-(84). These two conditions hold true irrespective of whether the IC constraint is binding or not.[25] Note that the profit functions of the problem with an IC constraint are in general different from those of the problem without IC constraint studied in Section 4.2 where there is no IC. However, we prove that the profit functions coincide when evaluated at the promised utilities $\underline{\omega}^*$ and $\bar{\omega}^*$.

The following proposition characterizes the equilibrium dynamics.

Proposition 9 *Suppose that the country starts in a recession, and is endowed with the initial promised utility ν.*

1. *If $\nu \geq \underline{\omega}^*$, then the IC is never binding, and the constrained optimal allocation, $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$, is identical to that in Proposition 6.*

2. *If $\nu < \underline{\omega}^*$, then there exist two thresholds, ϕ^* and $\tilde{\phi}(\nu)$, where $\phi^* = \tilde{\phi}(\underline{\omega}^*)$ (expressions in the proof in the appendix) such that:*

    (a) *If $\phi < \phi^*$, the PC is binding while the IC is not binding. The solution is not history-dependent and is determined as in Proposition 6 (in particular, $\underline{\omega}_\phi > \underline{\omega}^*$ and $p_\phi < p^*$).*

[22] The assumption that the deviating country is settled with the debt level b_0 irrespective of the history of the contract is made for simplicity. This keeps the outside option of the IC constant, avoiding the technical complications arising from an endogenous outside option.

[23] If the IC constraint (42) is not binding, the solution is as in Proposition 6. Note that, if the IC constraint is not binding at t, it will never bind in future, since the allocation of Proposition 6 entails a non-decreasing promised utility and a non-increasing effort path.

[24] The proof of the lemma is merged with the proof of Proposition 9.

[25] To see why the solution to (42)–(45) is unique, note that the concavity and monotonicity of $\underline{P}$ and $\bar{P}$ imply that Equation (44) determines a positive relationship between $\underline{\omega}$ and $\bar{\omega}$. Thus, Equation (45) yields an implicit decreasing relationship between p and $\underline{\omega}$, while (42) yields an implicit increasing relationship between p and $\underline{\omega}$.

(b) *If $\phi \in [\phi^*, \tilde{\phi}(\nu)]$, both the PC and the IC are binding. Effort and promised utilities are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ as given by Lemma 6. Consumption is determined by Equations (31) and (42) which yield:*

$$c^*_{\phi} = u^{-1}\left(\underline{\nu} - \phi - Z(b_0)\right). \tag{46}$$

Consumption and effort are lower than in the allocation of Proposition 6.

(c) *If $\phi > \tilde{\phi}(\nu)$, the IC is binding, while the PC is not binding. Effort and promised utilities are equal to $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. The consumption level is determined by the promise-keeping constraint (30). In particular, consumption is constant across ϕ and given by:*

$$c^*_{\tilde{\phi}(\nu)} = u^{-1}\left(\underline{\nu} - \tilde{\phi}(\nu) - Z(b_0)\right). \tag{47}$$

For given ν and ϕ, consumption and effort are lower than in the allocation of Proposition 6.

Consider an economy where, initially, $\nu < \underline{\omega}^*$ (recall that a low ν corresponds to a high initial debt in the decentralized equilibrium). If the first realization of ϕ is sufficiently low (case 2.a of Proposition 9), the IC is not binding, the allocation is not history-dependent, and the characterization of Proposition 6 applies. If the first realization of ϕ is larger than ϕ^*, the IC is binding, and Lemma 6 implies that effort and promised utility equal $(p^*, \underline{\omega}^*, \bar{\omega}^*)$. If $\phi \in [\phi^*, \tilde{\phi}(\nu)]$ (case 2.b), consumption is pinned down jointly by the PC and the promise-keeping constraint (consumption will then be decreasing in ϕ). Finally, if $\phi > \tilde{\phi}(\nu)$ (case 2.c) the PC imposes no constraint, and the initial consumption is determined only by the promise-keeping constraints. When the IC is binding (cases 2.b and 2.c), both consumption and effort are lower than in the second-best solution of Proposition 6. Intuitively, the planner cannot set effort at the efficient level due to the IC, and adjust optimally to the constraint by reducing current consumption and increasing promised utilities. Thus, the contract provides less consumption insurance than does the second-best constrained-efficient allocation. When the IC is binding, the promise utility increases from ν to $\underline{\omega}^*$. Thereafter, consumption, effort and promised utilities remain constant until a realization of ϕ lower than ϕ^* is observed. In summary, after one period the equilibrium is characterized as in the second best of Proposition 6.

Figure 5 is the analogue of Figure 4 in an economy in which the IC is binding in the initial period, i.e., $\nu < \underline{\omega}^*$. The left panel shows the dynamics of consumption and effort, whereas the right panel shows the dynamics of expected utility. The initial promised utility (ν) is consistent with a break-even condition for the planner, namely, $\underline{\Pi}(b) = \underline{P}(\nu)$. The dash-dotted line in Figure 5 is for comparison, and shows the second-best constrained-efficient allocation of Proposition 6 corresponding to the same sequence of ϕ's. In the first period, the realization of the stochastic process is in the range $\phi > \phi^*$. Thus, the IC is binding, and consumption and effort are below the second-best constrained-efficient level. After one period, consumption increases to meet the promise-keeping constraint, and remains constant (as do effort and promised utilities) thereafter until period seven, when the first realization in the range $\phi < \phi^*$ is observed. From that period onwards, the IC never binds again and the economy settles down to the (ex-post) constrained-efficient allocation. Note that the constraint that the reform effort must be self-enforcing reduces the country's ex-ante welfare. The reason is that, until period seven, the principal cannot extract the efficient reform effort level, and must offer the agent a lower consumption (compensated by a larger promised utility) to break even.[26]

[26] The left-hand figure does not show the initial ν. For instance, at time t=1, one can see $\underline{\omega}^*$ and $\bar{\omega}^*$, where, recall, $\nu < \underline{\omega}^*$. The initial ν is lower than the dashed black line. However, after one period, the promised utility is higher in the economy with the IC constraint than in the second-best.

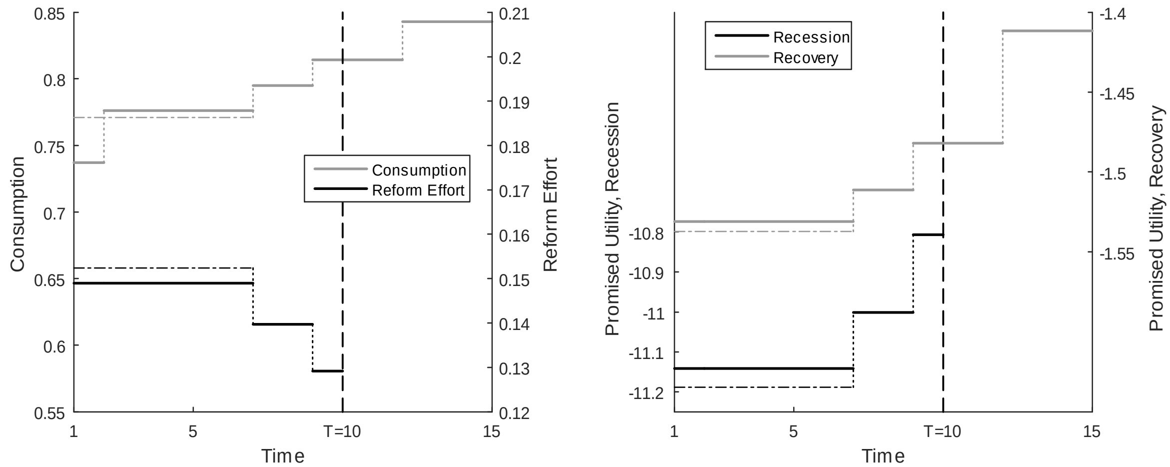

Figure 5: Simulation of consumption, effort, and promised utilities for a particular sequence of ϕ's where the IC is initially binding. Solid lines refer to an economy with an IC constraint. Dashed lines refer to the economy without an IC constraint.

Figure 5 yields simulated paths of consumption, effort and promised utility in the constrained optimal allocation for two otherwise identical economies where one economy (solid lines) is subject to the IC constraint, while the other economy (dashed lines) has no such constraints. The initial promised utility ν (not displayed) is lower than $\underline{\omega}^*$ implying that the IC is binding. In the first period, consumption and effort are lower in the economy with an IC constraint. In contrast, promised utility is higher. In other words, the planner provides less insurance by making consumption and effort initially lower, but growing at a higher speed. As of the second period, the dynamics of both economies are the same as in Figure 4.

6 Calibration

In this section, we study quantitative properties of the model. To this end, we calibrate the model economy to match some salient facts on sovereign debt. The main purpose of this exercise is to evaluate the welfare gain of going from the competitive equilibrium to the constrained optimal allocation. We will also be able to evaluate the welfare effect of various austerity programs. One common problem in the quantitative literature on sovereign debt is that it is difficult for these models to match observed values of debt-to-GDP ratios under realistic parameterizations (Arellano 2008; Yue 2010). As we will see, this is not a problem in our model. We will be able to match both default premia, recovery rates, and plausible debt-to-GDP ratios (possibly exceeding 200% during the recession).

A model period corresponds to one year. We normalize the GDP during normal times to $\bar{w} = 1$ and assume that the recession causes a drop in income of 38%, i.e., $\underline{w} = 0.62 \times \bar{w}$. This corresponds to the fall of GDP per capita for Greece between 2007 and 2013, relative to trend.[27] Since we focus on the return on government debt, the annual real gross interest rate is set to $R = 1.02$, implying

[27] GDP per capita of Greece fell from €18,924 to €14,551 between 2007 and 2013 (Eurostat). The annualized growth rate between 1997 and 2007 was 3.8%. The fall in output between 2007 and 2013 relative to trend is therefore 38%.

$\beta = 1/R = 0.98$. The risk aversion is set exogenously to $\gamma = 2$.

We assume a standard constant elasticity version of the effort cost function; $X(p) = \frac{\xi}{1+1/\varphi}(p)^{1+1/\varphi}$, where ξ regulates the average level of effort and φ regulates the elasticity of reform effort to changes in the return to reforms. We set the two parameters, φ and ξ, so as to match two points on the equilibrium effort function $\Psi(b)$. In particular, we assume that the effort at the debt limit is $\Psi(\bar{b}) = 10\%$, so that a country with a debt at the debt limit chooses an effort inducing an expected duration of the recession of one decade (we have Greece in mind). Moreover, we assume that the maximum effort is $\max_b \Psi(b) = 20\%$, inducing an expected recession duration of five years (we have Iceland and Ireland in mind). This implies setting $\varphi = 22.1$ and $\xi = 24.45$.

Finally, we determine the distribution of ϕ. To obtain a reasonable debt limit $\bar{b}$, we focus on a distribution with bounded support $[0, \bar{\phi}]$, where the maximum default cost realization is set so that the debt limit during normal times is $\bar{b}/\bar{w} = 180\%$. This implies $\bar{\phi} = 2.11$. The distribution of ϕ is a generalized Beta, with c.d.f. given by $F(\phi; \eta_1, \eta_2) = \mathcal{B}(\phi/\bar{\phi}, \eta_1, \eta_2)/\mathcal{B}(1, \eta_1, \eta_2)$, where $\mathcal{B}(x, \eta_1, \eta_2)$ denotes the incomplete Beta function $\mathcal{B}(x, \eta_1, \eta_2) = \int_0^x t^{\eta_1 - 1}(1-t)^{\eta_2 - 1}dt$. We set $\eta_1 = 0.8$ and $\eta_2 = 0.105$ so as to match two moments: an average post-renegotiation recovery rate of 62% (Tomz and Wright 2007) and an average default premium of 4% for a country which has a debt-output ratio of 100% during recession.[28] This was the average debt and average default premium for Greece, Ireland, Italy, Portugal, and Spain (GIIPS) during 2008-2012.[29]

We use the calibrated economy to evaluate the welfare gains of different policy arrangements. The welfare gains are measured as the equivalent variation in terms of an initial debt to output reduction in the market economy, namely, the reduction in initial debt required to make the borrower indifferent between staying in the market arrangement (with the reduction in debt) and moving to an alternative allocation.

We assume that the economy has an initial debt-output ratio of 100%, corresponding to $b_0 = 0.62$. We find that the welfare gain of going to the first best is larger than a one-time transfer of 100% of GDP (which is equivalent to forgiving all the outstanding debt). Similarly, the gain of going to the second best (which, as we know, can be implemented by an austerity program) is equivalent to a one-time transfer of 49% of GDP. Allowing for state-contingent debt, on the other hand, yields a mere 6% welfare gain, far smaller than the gain of moving to the second best. This shows that the moral hazard in the reform effort is responsible for the lion's share of the welfare loss of the market allocation relative to the second best.

7 Conclusions

This paper presents a theory of sovereign debt dynamics under limited commitment. A sovereign country issues debt to smooth consumption during a recession whose duration is uncertain and edogenous. The expected duration of the recession depends on the intensity of (costly) structural reforms. Both elements – the risk of repudiation and the need of structural reforms – are salient features of the recent European debt crisis.

[28] Our calibration is also in line with the findings of Reinhart and Trebesch (2014). They document an average debt relief of 40% of external government debt across both the 1930s and the 1980s/1990s. Moreover, the average debt relief is reported to be 21% of GDP for advanced economies in the 1930s, and 16% of GDP for emerging market economies in the 1980s/1990s. Even though we do not target this moment of the data in the calibration, our simulations yield an average debt relief of 22% of GDP, which is in the ball park of the estimates.

[29] In the simulations of the average recovery rate, we start the economy at the average debt-output ratio in 2008, which was 75% among the GIIPS countries.

The competitive equilibrium features repeated debt renegotiations. Renegotiations are more likely to occur during recessions and when the country has accumulated a high debt level. As a recession drags on, the country has an incentive to go deeper into debt. A higher debt level may in turn deter rather than stimulate economic reforms.

The theory bears normative predictions that are relevant for the management of the European crisis. The market equilibrium is inefficient for two reasons. On the one hand, the government of the sovereign country underinvests in structural reforms. The intuitive reason is that the short-run cost of reforms is entirely borne by the country, while their future benefits accrue in part to the creditors in the form of an *ex-post* increased price of debt, due to a reduction in the probability of renegotiation. On the other hand, the limited commitment to honor debt induces high risk premia and excess consumption volatility. A well-designed intervention of an international institution can improve welfare, as long as the institution can monitor the reform process. While we assume, for tractability, that the international institution can monitor reforms perfectly, our results carry over to a more realistic scenario where reforms are only imperfectly monitored. The optimal policy also entails an assistance program whereby an international organization provides the country with a constant transfer flow, deferring the repayment of debt to the time when the recession ends. The optimal contract factors in that this payment is itself subject to renegotiation risk.

A second implication is that, when the government of the indebted country credibly threatens to renege on an existing agreement, concessions should be made to avoid an outright repudiation. Contrary to a common perception among policy makers, a rigid commitment to enforce the terms of the original agreement is not optimal. Rather, the optimal policy entails the possibility of multiple renegotiations, which are reflected in the terms of the initial agreement.

To retain tractability, we make important assumptions that we plan to relax in future research. First, in our theory the default cost follows an exogenous stochastic process. In a richer model, this would be part of the equilibrium dynamics. Strategic delegation is a potentially important extension. In the case of Greece, voters may have an incentive to elect a radical government with the aim of delegating the negotiation power to an agent that has or perceives to have a lower default cost than have voters (cf. Rogoff 1985). In our current model, however, the stochastic process governing the creditor's outside option is exogenous, and is outside of the control of the government and creditors.

Second, again for simplicity, we assume that renegotiation is costless, that creditors can perfectly coordinate and that they have full bargaining power in the renegotiation game. Each of these assumptions could be relaxed. For instance, one can acknowledge that in reality the process of negotiation may entail costs. Moreover, as in the recent contention between Argentina and the so-called vulture funds, some creditors may hold out and refuse to accept a restructuring plan signed by a syndicate of lenders. Finally, the country may retain some bargaining power in the renegotiation. All these extensions would introduce interesting additional dimensions, and invalidate some of the strong efficiency results (for instance, the result that the competitive economy attains the second best in the absence of income fluctuations). However, we are confident that the gist of the results is robust to these extensions.

Finally, by focusing on a representative agent, we abstract from conflicts of interest between different groups of agents within the country. Studying the political economy of sovereign debt would be an interesting extension. We leave the exploration of these and other avenues to future work.

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9 Appendix

9.1 Proofs of lemmas, propositions, and corollaries.

Proof of Lemma 2. Assume $\bar{\Phi}$ is differentiable. Then, differentiating $b \times Q(b, \bar{w})$ with respect to b yields:

$$\begin{aligned}
\frac{d}{db}\{b \times Q(b,\bar{w})\} &= Q(b,\bar{w}) + b \times \frac{d}{db}\left(\frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right) + \frac{1}{R}\frac{1}{b}\int_0^{\bar{\Phi}(b)}\left(\bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right)\right) \\
&= Q(b,\bar{w}) - \frac{b}{R} f\left(\bar{\Phi}(b)\right)\bar{\Phi}'(b) + \\
&\quad \frac{b}{R}\frac{1}{b}\bar{\Phi}^{-1}\left(\bar{\Phi}(b)\right) \times f\left(\bar{\Phi}(b)\right) \times \bar{\Phi}'(b) - \underbrace{\frac{1}{R}\frac{1}{b}\int_0^{\bar{\Phi}(b)}\left(\bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right)}_{=Q(b,\bar{w}) - \frac{1}{R}\left(1-F\left(\bar{\Phi}(b)\right)\right)} \\
&= \frac{1}{R}\left(1 - F\left(\bar{\Phi}(b)\right)\right).
\end{aligned}$$

Since $F\left(\bar{\Phi}(b)\right) < 1$ for all $b < \bar{b}$, then $b \times Q(b, \bar{w})$ is monotone increasing in that range. The revenue from selling new bonds reaches a maximum at $b = \bar{b}$, since

$$\lim_{b \to b^-} \frac{d}{db}\{b \times Q(b,\bar{w})\} = \frac{1}{R}\left(1 - F\left(\bar{\Phi}\left(\bar{b}\right)\right)\right) = 0.$$

■

Proof of Proposition 1. The first order condition of (8) yields:

$$\frac{\frac{d}{db'}\{b' \times Q(b',\bar{w})\}}{Q(b',\bar{w}) \times b' + \bar{w} - \tilde{b}} + \frac{d}{db'}\beta EV\left(b', \bar{w}\right) = 0$$

The value function has a kink at $b = \hat{b}(\phi, \bar{w})$. Consider, first, the range where $b < \hat{b}(\phi, \bar{w})$. Differentiating the value function yields:

$$\frac{d}{db}V(b,\phi,\bar{w}) = -\frac{1}{Q(B(b,\bar{w}),\bar{w}) \times B(b,\bar{w}) + \bar{w} - b}.$$

Next, consider the region of renegotiation, $b > \hat{b}(\phi, \bar{w})$. In this case, $\frac{d}{db}V(b, \phi, w) = 0$.

Using the results above one obtains:

$$\begin{aligned}
\frac{d}{db}EV(b,\bar{w}) &= \int_0^{\Phi(b)} \frac{d}{db}V(b,\phi,\bar{w})\,dF(\phi) + \int_{\Phi(b)}^{\infty}\frac{d}{db}V(b,\phi,\bar{w})\,dF(\phi) \\
&= \int_{\bar{\Phi}(b)}^{\infty}\frac{-1}{Q(B(b,\bar{w}),\bar{w}) \times B(b,\bar{w}) + \bar{w} - b}dF(\phi) \\
&= -\frac{1 - F\left(\bar{\Phi}(b)\right)}{Q(B(b,\bar{w}),\bar{w}) \times B(b,\bar{w}) + \bar{w} - b} \qquad (48)
\end{aligned}$$

Plugging this expression back into the FOC, and leading the expression by one period, yields

$$\begin{aligned}
\frac{\frac{d}{db'}\left\{b' \times Q\left(b', \bar{w}\right)\right\}}{Q\left(b', \bar{w}\right) \times b' + \bar{w} - \tilde{b}} - \beta \frac{1 - F\left(\bar{\Phi}\left(b'\right)\right)}{Q\left(B\left(b', \bar{w}\right), \bar{w}\right) \times B\left(b', \bar{w}\right) + \bar{w} - b'} &= 0 \\
&\Rightarrow \\
\frac{Q\left(B\left(b', \bar{w}\right), \bar{w}\right) \times B\left(b', \bar{w}\right) + \bar{w} - b'}{Q\left(b', \bar{w}\right) \times b' + \bar{w} - \tilde{b}} &= \beta R \qquad (49)
\end{aligned}$$

where the last step uses the fact that $\frac{d}{db}\left\{b \times Q\left(b, \bar{w}\right)\right\} = \frac{1}{R}\left(1 - F\left(\bar{\Phi}\left(b\right)\right)\right)$, as shown in the proof of Lemma 2. The budget constraint, (1), left-hand of (49) is the consumption growth in case of repayment, so Equation (49) is equivalent to Equation (9) in the Proposition.

The second part of the Proposition follows from the observation that, in case of renegotiation, the same expression as (49) obtains, except that the numerator is $Q\left(B\left(b', \bar{w}\right), \bar{w}\right) \times B\left(b', \bar{w}\right) + \bar{w} - \hat{b}\left(\phi', \bar{w}\right)$, where, recall, $\hat{b}\left(\phi', \bar{w}\right) < b'$. Thus $c_{t+1}/c_t > \beta R$. ■

Lemma 7 *Assume $\beta R = 1$. Then, conditional on the realization ϕ, the equilibrium threshold debt that triggers default is given by:*

$$\begin{aligned}
\hat{b}\left(\phi, \bar{w}\right) &= \frac{\beta}{1 - \beta\left(1 - F\left(\phi\right)\right)} \int_0^{\phi} \hat{b}\left(x, \bar{w}\right) f\left(x\right) dx \qquad (50) \\
&+ \bar{w} \frac{1 - \exp\left(\beta \int_0^{\phi} x f\left(x\right) dx - \left(1 - \beta\left(1 - F\left(\phi\right)\right)\right) \times \phi\right)}{1 - \beta\left(1 - F\left(\phi\right)\right)}
\end{aligned}$$

Proof of Lemma 7. When $\beta R = 1$, one obtains:

$$\begin{aligned}
W^H\left(b, \bar{w}\right) &= \log\left(Q\left(b, \bar{w}\right) \times b + \bar{w} - b\right) \qquad (51) \\
&+ \left(1 - F\left(\bar{\Phi}\left(b\right)\right)\right) \times \beta W^H\left(b, \bar{w}\right) + \int_0^{\bar{\Phi}(b)} \beta W^H\left(\hat{b}\left(\phi, \bar{w}\right), \bar{w}\right) f\left(\phi\right) d\phi.
\end{aligned}$$

Evaluating $W^H\left(b, \bar{w}\right)$ at $\hat{b}\left(\phi, \bar{w}\right)$, and using Lemma 2 allows us to rewrite (51) as:

$$\begin{aligned}
W^H\left(\hat{b}\left(\phi, \bar{w}\right), \bar{w}\right) &= \log\left(Q\left(\hat{b}\left(\phi, \bar{w}\right), \bar{w}\right) \times \hat{b}\left(\phi, \bar{w}\right) + \bar{w} - \hat{b}\left(\phi, \bar{w}\right)\right) + \\
&\left(1 - F\left(\phi\right)\right) \times \beta W^H\left(\hat{b}\left(\phi, \bar{w}\right), \bar{w}\right) + \int_0^{\phi} \beta W^H\left(\hat{b}\left(x, \bar{w}\right), \bar{w}\right) f\left(x\right) dx
\end{aligned}$$

Recall that, given the realization ϕ, if $b = \hat{b}\left(\phi, \bar{w}\right)$, then the debtor is indifferent between renegotiating debt at the level $\hat{b}\left(\phi\right))$ and defaulting. Thus, $W^H\left(\hat{b}\left(\phi, \bar{w}\right), \bar{w}\right) = \log\left(\bar{w}\right) - \phi + \beta W^H\left(0, \bar{w}\right) = \frac{1}{1-\beta}\log\left(\bar{w}\right) - \phi$. The last equality follows from the fact that $W^H\left(0, \bar{w}\right) = \frac{1}{1-\beta}\log\left(\bar{w}\right)$, since when $b = 0$ and $\beta R = 1$, the country neither has an interest to default again, nor to accumulate any further debt. Using this condition to eliminate $W^H\left(b, \bar{w}\right)$ from (51) yields:

$$\begin{aligned}
\frac{1}{1-\beta}\log\left(\bar{w}\right) - \phi &= \log\left(Q\left(\hat{b}\left(\phi, \bar{w}\right), \bar{w}\right) \times \hat{b}\left(\phi, \bar{w}\right) + \bar{w} - \hat{b}\left(\phi, \bar{w}\right)\right) \\
&+ \left(1 - F\left(\phi\right)\right) \times \beta\left(\frac{1}{1-\beta}\log\left(\bar{w}\right) - \phi\right) + \int_0^{\phi} \beta\left(\frac{1}{1-\beta}\log\left(\bar{w}\right) - x\right) f\left(x\right) dx
\end{aligned}$$

Inverting the utility function, and simplifying terms, yields

$$\underbrace{Q\left(\hat{b}\left(\phi,\bar{w}\right),\bar{w}\right)\hat{b}\left(\phi,\bar{w}\right)}_{\text{new debt issuance}}+\underbrace{\bar{w}}_{\text{wage income}} = \underbrace{\hat{b}\left(\phi,\bar{w}\right)}_{\text{debt repayment}} + \bar{w}\exp\left[-\underbrace{\left(\beta\int_0^{\phi} xf\left(x\right)dx-\left(1-\beta\left(1-F\left(\phi\right)\right)\right)\times\phi\right)}_{\text{consumption under renegotiation}}\right]. \tag{52}$$

Next, evaluate the bond price, (5), at $b=\hat{b}\left(\phi\right)$ (recalling that $\beta R=1$), and substitute in the expression in (52). This yields:

$$\begin{aligned}&\hat{b}\left(\phi,\bar{w}\right)-\beta\left(1-F\left(\phi\right)\right)\times\hat{b}\left(\phi,\bar{w}\right)-\beta\int_0^{\phi}\hat{b}\left(x,\bar{w}\right)f\left(x\right)dx\\ =\;&\bar{w}-\bar{w}\exp\left(\beta\int_0^{\phi}xf\left(x\right)dx-\left(1-\beta\left(1-F\left(\phi\right)\right)\right)\times\phi\right)\end{aligned}$$

which in turn implies equation (50). ■

Proof of Proposition 2. Consider first the case of b close to zero, i.e., $b\in[0,b_1)$. Differentiate equation (12) with respect to b',

$$\begin{aligned}X''\left(\Psi\left(b'\right)\right)\Psi'\left(b'\right) &= \beta\left[\int_0^{\infty}\frac{\partial}{\partial b'}V\left(b',\phi',\bar{w}\right)dF\left(\phi\right)-\int_0^{\infty}\frac{\partial}{\partial b'}V\left(b',\phi',\underline{w}\right)dF\left(\phi\right)\right]\\ &= \beta\left[-\frac{1-F\left(\bar{\Phi}\left(b\right)\right)}{Q\left(B\left(b,\bar{w}\right),\bar{w}\right)\times B\left(b,\bar{w}\right)+\bar{w}-b}+\frac{1-F\left(\underline{\Phi}\left(b\right)\right)}{Q\left(B\left(b,\underline{w}\right)\right)\times B\left(b,\underline{w}\right)+\underline{w}-b}\right]\end{aligned} \tag{53}$$

Take the limit of equation (53) as $b'\to 0$

$$\lim_{b'\to 0}\left\{X''\left(\Psi\left(b'\right)\right)\Psi'\left(b'\right)\right\} = \lim_{b'\to 0}\left\{\beta\left[-\frac{1-F\left(\bar{\Phi}\left(b\right)\right)}{Q\left(B\left(b,\bar{w}\right),\bar{w}\right)\times B\left(b,\bar{w}\right)+\bar{w}-b}+\frac{1-F\left(\underline{\Phi}\left(b\right)\right)}{Q\left(B\left(b,\underline{w}\right)\right)\times B\left(b,\underline{w}\right)+\underline{w}-b}\right]\right\}$$
$$\Rightarrow$$

$$\begin{aligned}X''\left(\Psi\left(0\right)\right)\Psi'\left(0\right) &= \beta\left[-\frac{1-F\left(0\right)}{Q\left(B\left(0,\bar{w}\right),\bar{w}\right)\times B\left(0,\bar{w}\right)+\bar{w}-0}+\frac{1-F\left(0\right)}{Q\left(B\left(0,\underline{w}\right)\right)\times B\left(0,\underline{w}\right)+\underline{w}-0}\right]\\ &= \beta\left(1-F\left(0\right)\right)\left[-\frac{1}{\bar{w}}+\frac{1}{Q\left(B\left(0,\underline{w}\right)\right)\times B\left(0,\underline{w}\right)+\underline{w}-0}\right],\end{aligned} \tag{54}$$

where the last equation uses the fact that during normal times $c=\bar{w}$ if $b=0$. Note that during recession, the annualized present value of income is strictly smaller than $\bar{w}$. Therefore, it can never be optimal to choose consumption during recession larger than or equal to $\bar{w}$ when $b=0$. Since the marginal utility of consumption is larger in a recession than during normal times, the right-hand side of equation (54) must be strictly positive. Since $X''>0$, it must be that $\lim_{b\to 0}\Psi'\left(b\right)=\Psi'\left(0\right)>0$. By continuity it follows that $\Psi'\left(b\right)$ will be positive for a range of b close to $b=0$, so there must exist a $b_1>0$ such that $\Psi'\left(b\right)>0$ for all $b\in[0,b_1)$.

Consider, next, the case when $b \in \left[\bar{b}^R, \bar{b}\right)$, in which case $F\left(\underline{\Phi}(b)\right) = 1$ and $F\left(\bar{\Phi}(b)\right) < 1$. This implies that equation (53) can be written as

$$X''\left(\Psi\left(b'\right)\right)\Psi'\left(b'\right) = -\beta\frac{1-F\left(\bar{\Phi}(b)\right)}{Q\left(B\left(b,\bar{w}\right),\bar{w}\right)\times B\left(b,\bar{w}\right)+\bar{w}-b} < 0,$$

which establishes that $\Psi'(b) < 0$ for all $b \in \left[\bar{b}^R, \bar{b}\right)$ and with strict inequality also for $b = \bar{b}^R$. By continuity it follows that there exists a $b_2 < \bar{b}^R$ such that $\Psi'(b) < 0$ for all $b \in \left(b_2, \bar{b}\right)$. Finally, for $b \geq \bar{b}$, $F\left(\underline{\Phi}(b)\right) = F\left(\bar{\Phi}(b)\right) = 1$ so the right-hand side of equation (53) becomes zero, implying that $\Psi'(b) = 0$. ■

Proof of Lemma 4. Differentiating the bond revenue with respect to b yields

$$\begin{aligned}
\frac{d}{db}\left\{Q\left(b,\underline{w}\right)b\right\} &= \frac{d}{db}\left\{pbQ\left(b,\bar{w}\right)+(1-p)\,b\hat{Q}\left(b,\underline{w}\right)\right\}+\Psi'(b)\times\left(Q\left(b,\bar{w}\right)-\hat{Q}\left(b,\underline{w}\right)\right)b \\
&= \Psi(b)\times\frac{1}{R}\left(1-F\left(\bar{\Phi}(b)\right)\right)+(1-\Psi(b))\times\frac{1}{R}\left(1-F\left(\underline{\Phi}(b)\right)\right) \\
&\quad +\Psi'(b)\times\left(Q\left(b,\bar{w}\right)-\hat{Q}\left(b,\underline{w}\right)\right)b,
\end{aligned} \tag{55}$$

where the second equality can be derived as following:

$$\begin{aligned}
\frac{d}{db}\left\{b\times Q\left(b,\underline{w}\right)\right\} &= \frac{d}{db}\left\{pbQ\left(b,\bar{w}\right)+(1-p)\,b\hat{Q}\left(b,\underline{w}\right)\right\} \\
&= p\frac{1}{R}\left(1-F\left(\bar{\Phi}(b)\right)\right)+(1-p)\,\hat{Q}\left(b,\underline{w}\right) \\
&\quad +(1-p)\left[\begin{array}{c} -\frac{b}{R}f\left(\underline{\Phi}(b)\right)\times\underline{\Phi}'(b)- \\ \frac{1}{R}\frac{1}{b}\int_0^{\underline{\Phi}(b)}\left(\underline{\Phi}^{-1}(\phi)\times f(\phi)\ d\phi\right)+\frac{1}{R}bf\left(\underline{\Phi}(b)\right)\underline{\Phi}'(b)\end{array}\right] \\
&= p\frac{1}{R}\left(1-F\left(\bar{\Phi}(b)\right)\right)+(1-p)\,\frac{1}{R}\left(1-F\left(\underline{\Phi}(b)\right)\right)
\end{aligned}$$

Consider, first, the case in which $\Psi(b) = p$. $\Psi'(b) = 0$. In this case, debt revenue is increase for all $b < \bar{b}$, since, then, $p/R\times\left(1-F\left(\bar{\Phi}(b)\right)\right)+(1-p)/R\times\left(1-F\left(\underline{\Phi}(b)\right)\right) > 0$. Moreover, it reaches a maximum at $b = \bar{b}$ (recall that $F\left(\bar{\Phi}(b)\right) < F\left(\underline{\Phi}(b)\right)$ for all $b < \bar{b}$). This establishes that, if $\Psi(b) = p$, then $\bar{b}^R = \bar{b}$.

Consider, next, the general case. Proposition 2 implies that, in the range where $b \in [b_2, \bar{b}]$, $\Psi'(b) < 0$ $\Psi'\left(\bar{b}\right) < 0$. Since $Q\left(b,\bar{w}\right) > \hat{Q}\left(b,\underline{w}\right)$, then, in a left neighborhood of $\bar{b}$, $\Psi'(b)\times\left[Q\left(b,\bar{w}\right)-\hat{Q}\left(b,\underline{w}\right)\right]b < 0$. This means that, starting from $\bar{b}$, it is possible to increase the debt revenue by reducing debt. Hence, $\bar{b}^R = \bar{b}$. ■

Proof of Lemma 3. If the country can, in the initial period only, contract on effort when issuing new debt, the problem becomes

$$\begin{aligned}
&\max_{b',p^*}\left\{\ln\left(Q\left(b',\underline{w}\right)\times b'+\underline{w}-\tilde{b}\right)-X\left(p^*\right)\right. \\
&\left.+\beta p^*\times EV\left(b',\bar{w}\right)+\beta\left(1-p^*\right)\times EV\left(b',\underline{w}\right)\right\}.
\end{aligned}$$

Note that the next-period value function V is the same as in the standard problem, since we consider a one-period deviation. The first-order condition with respect to p becomes

$$\begin{aligned} 0 &= \frac{\frac{d}{dp^*}\left\{Q\left(b',\underline{w}\right)b'\right\}}{Q\left(b',\underline{w}\right)\times b'+\underline{w}-\tilde{b}} - X'\left(p^*\right)+\beta\left(EV\left(b',\bar{w}\right)-EV\left(b',\underline{w}\right)\right) \\ &\Rightarrow \\ X'\left(p^*\right) &= \frac{\left[Q\left(b',\bar{w}\right)-\hat{Q}\left(b',\underline{w}\right)\right]b'}{Q\left(b',\underline{w}\right)\times b'+\underline{w}-\tilde{b}}+\beta\left[\int_0^\infty V\left(b',\phi',\bar{w}\right)dF\left(\phi\right)-\int_0^\infty V\left(b',\phi',\underline{w}\right)dF\left(\phi\right)\right] \\ &> \beta\left[\int_0^\infty V\left(b',\phi',\bar{w}\right)dF\left(\phi\right)-\int_0^\infty V\left(b',\phi',\underline{w}\right)dF\left(\phi\right)\right], \end{aligned} \tag{56}$$

where the last equation follows from the fact that $Q\left(b',\bar{w}\right)>\hat{Q}\left(b',\underline{w}\right)$ and

$$\begin{aligned} \frac{d}{dp}\left\{Q\left(b',\underline{w}\right)b'\right\} &= \frac{d}{dp}\left\{\left[pQ\left(b',\bar{w}\right)+(1-p)\hat{Q}\left(b',\underline{w}\right)\right]b'\right\} \\ &= \left[Q\left(b',\bar{w}\right)-\hat{Q}\left(b',\underline{w}\right)\right]b' \end{aligned}$$

The right-hand side of the inequality in equation (56) is the optimal effort in the standard case, given in equation (12). This establishes the lemma. ■

Proof of Proposition 3. The procedure is analogous to the derivation of the CEE in normal times. The first order condition of (15) yields

$$\begin{aligned} 0 &= \frac{\frac{d}{db'}\left\{Q\left(b',\underline{w}\right)\times b'\right\}}{Q\left(b',\underline{w}\right)\times b'+\underline{w}-\tilde{b}}+\beta\left[1-\Psi\left(b'\right)\right]\frac{d}{db'}EV\left(b',\underline{w}\right)+\beta\Psi\left(b'\right)\frac{d}{db'}EV\left(b',\bar{w}\right) \\ &\quad +\frac{d\Psi\left(b'\right)}{db'}\underbrace{\left(-X'\left(\Psi\left(b'\right)\right)+\beta\left[EV\left(b',\bar{w}\right)-EV\left(b',\underline{w}\right)\right]\right)}_{=0\text{ due to an envelope argument}}. \end{aligned}$$

The value function has a kink at $b=\hat{b}\left(\phi,\underline{w}\right)$. In the range where $b<\hat{b}\left(\phi,\underline{w}\right)$,

$$\frac{d}{db}V\left(b,\phi,\underline{w}\right)=-\frac{1}{Q\left(B\left(b,\underline{w}\right),\underline{w}\right)\times B\left(b,\underline{w}\right)+\underline{w}-b},$$

while in the range where $b>\hat{b}\left(\phi,w\right)$, $\frac{d}{db}V\left(b,\phi,\underline{w}\right)=0$. Moreover:

$$\frac{d}{db}EV\left(b,\underline{w}\right)=-\frac{1-F\left(\underline{\Phi}\left(b\right)\right)}{Q\left(B\left(b,\underline{w}\right)\right)\times B\left(b,\underline{w}\right)+\underline{w}-b}$$

Plugging this back into the FOC (after leading the expression by one period) yields the CEE

$$\begin{aligned} 0 &= \frac{\Psi\left(b'\right)\times\left(1-F\left(\bar{\Phi}\left(b'\right)\right)\right)+\left[1-\Psi\left(b'\right)\right]\times\left(1-F\left(\underline{\Phi}\left(b'\right)\right)\right)}{Q\left(b',\underline{w}\right)\times b'+\underline{w}-\tilde{b}} \\ &\quad +\frac{\Psi'\left(b'\right)\times R\left[Q\left(b',\bar{w}\right)-\hat{Q}\left(b',\underline{w}\right)\right]b'}{Q\left(b',\underline{w}\right)\times b'+\underline{w}-\tilde{b}} \\ &\quad -\beta R\left(\begin{array}{c}\left[1-\Psi\left(b'\right)\right]\frac{1-F\left(\underline{\Phi}\left(b'\right)\right)}{Q\left(B\left(b',\underline{w}\right)\right)\times B\left(b',\underline{w}\right)+\underline{w}-b'}+ \\ +\Psi\left(b'\right)\frac{1-F\left(\bar{\Phi}\left(b'\right)\right)}{Q\left(B\left(b',\bar{w}\right)\right)\times B\left(b',\bar{w}\right)+\bar{w}-b'}\end{array}\right), \end{aligned}$$

where the equality follows from Lemma 4. Rearranging terms yields

$$\begin{aligned} & \beta R \left(\begin{array}{c} \frac{[1-\Psi(b')]\times[1-F(\underline{\Phi}(b'))]}{\Psi(b')\times\left(1-F\left(\bar{\Phi}(b')\right)\right)+[1-\Psi(b')]\times(1-F(\underline{\Phi}(b')))} \frac{Q(b',\underline{w})\times b'+\underline{w}-\tilde{b}}{Q(B(b',\underline{w}))\times B(b',\underline{w})+\underline{w}-b'} + \\ + \frac{\Psi(b')\times\left[1-F\left(\bar{\Phi}(b')\right)\right]}{\Psi(b')\times\left(1-F\left(\bar{\Phi}(b')\right)\right)+[1-\Psi(b')]\times(1-F(\underline{\Phi}(b')))} \frac{Q(b',\underline{w})\times b'+\underline{w}-\tilde{b}}{Q(B(b',\bar{w}))\times B(b',\bar{w})+\bar{w}-b'} \end{array} \right) \\ = & \; 1 + \frac{\Psi'(b')}{\Psi(b')\left\{\left(1-F\left(\bar{\Phi}(b')\right)\right) + [1-\Psi(b')]\times\left(1-F\left(\underline{\Phi}(b')\right)\right)\right\}} R\left[Q(b',\bar{w}) - \hat{Q}(b',\underline{w})\right] b', \end{aligned} \tag{57}$$

which is the same expression as in (17). ■

Proof of Lemma 5. Write the Lagrangian (with the associated multiplier, λ):

$$\max_{c,p} \frac{1}{1-\beta(1-p)}(\underline{w}-c) + \frac{\beta}{1-\beta}\frac{p}{1-\beta(1-p)}(\bar{w}-c) + \lambda\left(\frac{u(c)}{1-\beta} - \frac{1}{1-\beta(1-p)}X(p) - \nu\right)$$

Differentiating with respect to c yields the standard condition:

$$\frac{1}{\lambda} = u'(c).$$

We can now substitute this condition into the program, and maximize over p (after eliminating the Lagrange multiplier):

$$\max_{p} \frac{1}{1-\beta(1-p)}(\underline{w}-c) + \frac{\beta}{1-\beta}\frac{p}{1-\beta(1-p)}(\bar{w}-c) + \frac{1}{u'(c)}\left(\frac{u(c)}{1-\beta} - \frac{1}{1-\beta(1-p)}X(p) - \nu\right).$$

The first-order condition yields

$$\begin{aligned} 0 \;\; = \;\; & -\frac{\beta}{[1-\beta(1-p)]^2}(\underline{w}-c) + \frac{\beta}{1-\beta}\frac{1-\beta}{[1-\beta(1-p)]^2}(\bar{w}-c) \\ & + \frac{1}{u'(c)}\frac{\beta}{[1-\beta(1-p)]^2}X(p) - \frac{1}{u'(c)}\frac{1}{1-\beta(1-p)}X'(p). \end{aligned}$$

Simplifying terms yields equation (20). ■

Proof of Proposition 4. We write the Lagrangian,

$$\begin{aligned} \bar{\Lambda} \;\; = \;\; & \int_{\aleph} [\bar{w} - c_\phi + \beta P(\bar{\omega}_\phi, \bar{w})]\, dF(\phi) + \bar{\mu}\left(\int_{\aleph} [u(c_\phi) + \beta\bar{\omega}_\phi]\, dF(\phi) - v\right) \\ & + \int_{\aleph} \bar{\lambda}_\phi [u(c_\phi) + \beta\bar{\omega}_\phi - \bar{v} + \phi]\, d\phi, \end{aligned}$$

with the associated multipliers $\bar{\mu}$ and $\bar{\lambda}_\phi$ (the notation $\underline{\mu}$ and $\underline{\lambda}_\phi$ will denote the corresponding multipliers in recession). The first-order conditions yield

$$f(\phi) = u'(c_\phi)\left(\bar{\mu} f(\phi) + \underline{\lambda}_\phi\right), \tag{58}$$

$$\bar{\lambda}_\phi + \bar{\mu} f(\phi) = -\bar{P}'(\bar{\omega}_\phi)\, f(\phi). \tag{59}$$

The envelope condition yields

$$-\bar{P}'(\nu) = \bar{\mu} \tag{60}$$

The two first-order conditions and the envelope condition jointly imply that

$$u'(c_\phi) = -\frac{1}{\bar{P}'(\bar{\omega}_\phi)} \tag{61}$$

$$\bar{P}'(\bar{\omega}_\phi) = \bar{P}'(\nu) - \frac{\lambda_\phi}{f(\phi)}. \tag{62}$$

Note that (61) is equivalent to (26) in the text. Consider, next, two cases, namely, when the PC is binding ($\underline{\lambda}_\phi > 0$) and then it is not binding ($\underline{\lambda}_\phi = 0$).

When the Participation Constraint is binding, $\underline{\lambda}_\phi > 0$. (62) implies then that $\bar{\omega}_\phi > \nu$. Then, (61) and (27) determine jointly the solution for $(c_\phi, \bar{\omega}_\phi)$. When the Participation Constraint is not binding, $\underline{\lambda}_\phi = 0$. (62) implies then that $\bar{\omega}_\phi = \nu$ and $c_\phi = c(\nu)$. ■

Proof of Proposition 5. We prove the proposition by deriving a contradiction. To this aim, suppose that, for $\bar{\Pi}(b) = \bar{P}(\nu)$, the planner can deliver more utility to the agent than can the competitive equilibrium. Namely, $\nu > EV(b, \bar{w})$. Then, since $\bar{P}$ is a decreasing strictly concave function, we must have that $\bar{P}(EV(b, \bar{w})) > \bar{P}(\nu)$ and $\bar{P}'(EV(b, \bar{w})) > \bar{P}'(\nu)$. We show that this inequality, along with the set of optimality conditions, induces a contradiction.

First, recall, that equation (5) implies that $\bar{\Pi}(b) = RQ(b, \bar{w})\, b$. Thus,

$$\bar{P}(EV(b, \bar{w})) > \bar{P}(\nu) = RQ(b, \bar{w})\, b, \tag{63}$$

where $EV(b, \bar{w})$ is decreasing in b. Differentiating the two sides of the inequality (63) with respect to b yields

$$\begin{aligned} & \bar{P}'(EV(b, \bar{w})) \times \frac{d}{db} EV(b, \bar{w}) \\ > \; & \frac{d}{db}[Q(b, \bar{w})\, b] \times R = 1 - F\left(\bar{\Phi}(b)\right), \end{aligned} \tag{64}$$

where the right-hand side equality follows from the proof of Lemma 2. Next, equation (48) implies that

$$\begin{aligned} \frac{d}{db} EV(b, \bar{w}) &= -\frac{1 - F\left(\bar{\Phi}(b)\right)}{Q(B(b, \bar{w}), \bar{w}) \times B(b, \bar{w}) + \bar{w} - b} \\ &= -\left[1 - F\left(\bar{\Phi}(b)\right)\right] \times u'\left(C^H(b, \bar{w})\right), \end{aligned}$$

where $C^H(b, \bar{w}) = Q(B(b, \bar{w}), \bar{w}) \times B(b, \bar{w}) + \bar{w} - b$ is the consumption level in the competitive equilibrium when the debt b is honored. Plugging in the expression of $\frac{d}{db} EV(b, \bar{w})$ allows us to simplify (64) as follows:

$$u'\left(C^H(b, \bar{w})\right) > -\frac{1}{\bar{P}'\left(EV\left(\bar{b}, \bar{w}\right)\right)} \tag{65}$$

Next, note that $C^H(b, \bar{w}) = c(\nu)$. Equation (65) yields $u'(c(\nu)) > -\frac{1}{\bar{P}'\left(EV\left(\bar{b}, \bar{w}\right)\right)}$, while (61) yields that $u'(c(\nu)) = -\frac{1}{\bar{P}'(\nu)}$. Thus, the two conditions jointly implies that $-\frac{1}{\bar{P}'(\nu)} > -\frac{1}{\bar{P}'\left(EV\left(\bar{b}, \bar{w}\right)\right)}$ which in turn implies that $\nu < EV\left(\bar{b},\bar{w}\right)$, since $\bar{P}$ is decreasing and concave. This contradicts the assumption that $\nu > EV\left(\bar{b},\bar{w}\right)$.

The analysis thus far implies that $\nu \leq EV\left(\bar{b},\bar{w}\right)$. However, that $\nu < EV\left(b,\bar{w}\right)$ can also be safely ruled out, because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu = EV\left(b,\bar{w}\right)$. ■

Proof of Proposition 6. We write the Lagrangian,

$$\begin{aligned}\underline{\Lambda} &= \int_{\aleph}\left[\underline{w} - c_\phi + \beta\left((1-p_\phi)\,\underline{P}\left(\underline{\omega}_\phi\right) + p_\phi \bar{P}\left(\bar{\omega}_\phi\right)\right)\right] dF(s) \\ &\quad + \underline{\mu}\left(\int_{\aleph}\left(u\left(c_\phi\right) - X\left(p_\phi\right) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right)\right) dF(\phi) - v\right) \\ &\quad + \int_{\aleph} \underline{\lambda}_\phi\left(u\left(c_\phi\right) - X\left(p_\phi\right) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi \omega_\phi\right) - \underline{\nu} + \phi\right) ds.\end{aligned}$$

The first-order conditions yield:

$$f(\phi) = u'(c_s)\left(\underline{\mu}\, f(\phi) + \underline{\lambda}_\phi\right) \tag{66}$$

$$\underline{\lambda}_\phi + \underline{\mu}\, f(\phi) = -\underline{P}'\left(\underline{\omega}_\phi\right)\, f(\phi) \tag{67}$$

$$\underline{\lambda}_\phi + \underline{\mu}\, f(\phi) = -\bar{P}'\left(\bar{\omega}_\phi\right)\, f(\phi) \tag{68}$$

$$\beta\left(\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right)\, f(\phi) = \left(\underline{\lambda}_\phi + \underline{\mu}\, f(\phi)\right)\left(X'\left(p_\phi\right) - \beta\left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right) \tag{69}$$

The envelope condition yields:

$$-P'(\nu) = -\underline{\mu} \tag{70}$$

Combining the first-order conditions and the envelope condition yields:

$$\begin{aligned} u'(c_s) &= -\frac{1}{\underline{P}'\left(\underline{\omega}_\phi\right)} \qquad (71) \\ \underline{P}'\left(\underline{\omega}_\phi\right) &= \underline{P}'(\nu) - \frac{\underline{\lambda}_\phi}{f(\phi)} \\ \underline{P}'\left(\underline{\omega}_\phi\right) &= \bar{P}'\left(\bar{\omega}_\phi\right) \\ X'\left(p_\phi\right) &= \beta\left(u'\left(c_\phi\right)\left(\bar{P}\left(\bar{\omega}_\phi\right) - \underline{P}\left(\underline{\omega}_\phi\right)\right) + \left(\bar{\omega}_\phi - \underline{\omega}_\phi\right)\right)\end{aligned}$$

We distinguish two cases, namely, when the PC is binding ($\underline{\lambda}_\phi > 0$) and then it is not binding ($\underline{\lambda}_\phi = 0$).

(I) When the Participation Constraint is binding and the recession continues, $\underline{\lambda}_\phi > 0$, $\underline{\omega}_\phi > \nu$, and

$$u\left(c_\phi\right) - X\left(p_\phi\right) + \beta\left((1-p_\phi)\,\underline{\omega}_\phi + p_\phi\bar{\omega}_\phi\right) = \underline{\nu} - \phi_\phi$$

Then, (32), (34), (35) and (33) determine jointly the solution for $\left(c_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi, p_\phi\right)$. As usual, there is in this case no history dependence, i.e., v does not matter. [If the recession ends, and $\bar{\omega}_\phi > \nu$, then $u\left(c_\phi\right) + \beta\bar{\omega}_\phi = \bar{\nu} - \phi_\phi$]

(II) When the Participation Constraint is not binding, $\underline{\lambda}_\phi = 0$. Then, $\underline{\omega}_\phi = \nu$ and $c_s = \underline{c}(\nu)$. The solution is history dependent. Moreover, during a recession consumption is either constant (if $\lambda_\phi = 0$) or increasing (if $\lambda_\phi > 0$). Moreover, (35) implies that

$$\beta\left(u'\left(\underline{c}(\nu)\right)\left(\bar{P}\left(\bar{\omega}(\nu)\right) - \underline{P}(\nu)\right) + \left(\bar{\omega}(\nu) - \nu\right)\right) = X'\left(p(\nu)\right), \tag{72}$$

namely, the planner requires constant effort over the set of states for which the constraint is not binding: $p_s = p(\nu)$. Differentiating the left-hand side yields

$$
\begin{aligned}
& \underbrace{u''\left(\underline{c}(\nu)\right)\underline{c}'(\nu) \times \left(P\left(\bar{\omega}(\nu), \bar{w}\right) - \underline{P}(\nu)\right)}_{<0} + \left(u'\left(\underline{c}(\nu)\right)\underline{P}'(\nu) + 1\right)\left(\bar{\omega}'(\nu) - 1\right) \\
= \; & u''\left(\underline{c}(\nu)\right)\underline{c}'(\nu) \times \left(\bar{P}\left(\bar{\omega}(\nu)\right) - \underline{P}(\nu)\right) < 0
\end{aligned}
$$

since, recall, (32) implies that $\frac{d}{d\nu}P(\nu, \underline{w}) = -1/u'\left(\underline{c}(\nu)\right)$. This implies that the right-hand side must also be decreasing in ν. Since X is concave an increasing, this implies in turn that $p(\nu)$ must be increasing in ν.

(III) Finally, we prove that $\underline{\omega}_\phi < \bar{\omega}_\phi$, i.e., conditional on ϕ the planner promises a higher continuation utility if the economy recovers than it remains in recession. To this aim, note that it is more expensive for the planner to deliver a given promised utility during a recession than in normal times. Thus, $\underline{\mu} > \bar{\mu}$ (where μ is the marginal cost to the planner of promising a certain utility level). Hence, the respective envelope conditions, (60) and (70), imply that, for any x,

$$\underline{P}'(x) < \bar{P}'(x).$$

Next that, since both $P(x, \underline{w})$ and $P(x, \bar{w})$ are decreasing concave functions of x, then

$$\underline{P}'(x_1) = \bar{P}'(x_2) \Leftrightarrow x_1 < x_2.$$

Finally, we have established that $\underline{P}'\left(\underline{\omega}_\phi\right) = \bar{P}'\left(\bar{\omega}_\phi\right)$ (see proof of Proposition 6). Thus, $\underline{\omega}_\phi < \bar{\omega}_\phi$. ■

Proof of Proposition 7. We first derive the CEEs (i), and then show that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$ (ii).

(i) The first order conditions with respect to $b'_{\bar{w}}$ and $b'_{\underline{w}}$ in problem (36) yields

$$
\begin{aligned}
0 &= \frac{\frac{d}{db'_{\bar{w}}}\left\{b'_{\underline{w}} \times Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + b'_{\bar{w}} \times Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right\}}{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \tilde{b}} + \beta\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\frac{d}{db'_{\bar{w}}}EV\left(b'_{\bar{w}}, \bar{w}\right) \\
& \quad + \frac{d\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{db'_{\bar{w}}}\underbrace{\left[-X'\left(\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right) + \beta\left(EV\left(b'_{\bar{w}}, \bar{w}\right) - EV\left(b'_{\underline{w}}, \underline{w}\right)\right)\right]}_{=0 \text{ by the envelope theorem}}, \\
0 &= \frac{\frac{d}{db'_{\underline{w}}}\left\{b'_{\underline{w}} \times Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) + b'_{\bar{w}} \times Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right\}}{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \tilde{b}} + \beta\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right]\frac{d}{db'_{\underline{w}}}EV\left(b'_{\underline{w}}, \underline{w}\right),
\end{aligned}
$$

where $\tilde{b} = \min\left\{b_{\underline{w}}, \hat{b}(\phi, \underline{w})\right\}$. The value function has a kink at $b_{\underline{w}} = \hat{b}(\phi, \underline{w})$. Consider, first, the range where repayment is optimal, $b_{\underline{w}} < \hat{b}(\phi, \underline{w})$. Differentiating the value function yields:

$$
\begin{aligned}
\frac{d}{db}V(b, \phi, \bar{w}) &= -\frac{1}{Q\left(B(b, \bar{w}), \bar{w}\right) \times B(b, \bar{w}) + \bar{w} - b}, \\
\frac{d}{db}V(b, \phi, \underline{w}) &= -\frac{1}{Q_{\underline{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\underline{w}}(b) + Q_{\bar{w}}\left(B_{\underline{w}}(b), B_{\bar{w}}(b)\right) \times B_{\bar{w}}(b) + \underline{w} - b}
\end{aligned}
$$

where $B_{\underline{w}}$ and $B_{\bar{w}}$ denote the optimal issuance of the two assets, respectively. Next, consider the region of renegotiation, $b > \hat{b}(\phi, \underline{w})$. In this case, $\frac{d}{db}V(b, \phi, \underline{w}) = 0$.

In analogy with equation (48), we obtain:

$$\begin{aligned}
\frac{d}{db_{\underline{w}}}EV\left(b_{\underline{w}},\underline{w}\right) &= \int_0^{\underline{\Phi}\left(b_{\underline{w}}\right)} \frac{d}{db_{\underline{w}}}V\left(b_{\underline{w}},\phi,\underline{w}\right)dF\left(\phi\right) + \int_{\underline{\Phi}\left(b_{\underline{w}}\right)}^{\infty} \frac{d}{db_{\underline{w}}}V\left(b_{\underline{w}},\phi,\underline{w}\right)dF\left(\phi\right) \\
&= 0 - \int_{\underline{\Phi}\left(b_{\underline{w}}\right)}^{\infty} \frac{1}{\left\{\begin{array}{c} Q_{\underline{w}}\left(B_{\underline{w}}\left(b_{\underline{w}}\right),B_{\bar{w}}\left(b_{\underline{w}}\right)\right)\times B_{\underline{w}}\left(b_{\underline{w}}\right)+ \\ Q_{\bar{w}}\left(B_{\underline{w}}\left(b_{\underline{w}}\right),B_{\bar{w}}\left(b_{\underline{w}}\right)\right)\times B_{\bar{w}}\left(b_{\underline{w}}\right)+\underline{w}-b_{\underline{w}}\end{array}\right\}} f\left(\phi\right)d\phi \\
&= -\frac{1-F\left(\underline{\Phi}\left(b_{\underline{w}}\right)\right)}{Q_{\underline{w}}\left(B_{\underline{w}}\left(b_{\underline{w}}\right),B_{\bar{w}}\left(b_{\underline{w}}\right)\right)\times B_{\underline{w}}\left(b_{\underline{w}}\right)+Q_{\bar{w}}\left(B_{\underline{w}}\left(b_{\underline{w}}\right),B_{\bar{w}}\left(b_{\underline{w}}\right)\right)\times B_{\bar{w}}\left(b_{\underline{w}}\right)+\underline{w}-b_{\underline{w}}}
\end{aligned} \tag{73}$$

Plugging these expressions back into the FOC, and leading the expressions by one period, yields

$$\begin{aligned}
& \frac{\frac{d}{db'_{\bar{w}}}\left\{b'_{\underline{w}}\times Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)+b'_{\bar{w}}\times Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right\}}{Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\underline{w}}+Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\bar{w}}+\underline{w}-\tilde{b}} \\
= \;& \beta\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\frac{1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)}{Q\left(B\left(b'_{\bar{w}},\bar{w}\right),\bar{w}\right)\times B\left(b'_{\bar{w}},\bar{w}\right)+\bar{w}-b'_{\bar{w}}} \\
& \frac{\frac{d}{db'_{\underline{w}}}\left\{b'_{\underline{w}}\times Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)+b'_{\bar{w}}\times Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right\}}{Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\underline{w}}+Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\bar{w}}+\underline{w}-\tilde{b}} \\
= \;& \beta\left[1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right]\frac{1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)}{Q_{\underline{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right),B_{\bar{w}}\left(b'_{\underline{w}}\right)\right)\times B_{\underline{w}}\left(b'_{\underline{w}}\right)+Q_{\bar{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right),B_{\bar{w}}\left(b'_{\underline{w}}\right)\right)\times B_{\bar{w}}\left(b'_{\underline{w}}\right)+\underline{w}-b'_{\underline{w}}}
\end{aligned}$$

The marginal revenues from issuing each security (including price externalities for the other security) are given by, respectively:

$$\begin{aligned}
& \frac{d}{db'_{\underline{w}}}\left\{b'_{\underline{w}}\times Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)+b'_{\bar{w}}\times Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right\} \\
= \;& \frac{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)-\frac{\frac{\partial}{\partial b'_{\underline{w}}}\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\underline{w}}+b'_{\bar{w}}\times\frac{\partial}{\partial b'_{\underline{w}}}Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right) \\
= \;& \frac{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)+\frac{\partial\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}}\times\left(\frac{Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}-\frac{Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\underline{w}}}{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}\right) \\
= \;& \frac{1-\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(1-F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)+\underbrace{\frac{\partial\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{\partial b'_{\underline{w}}}}_{>0}\times\underbrace{\Delta\left(b'_{\underline{w}},b'_{\bar{w}}\right)}_{>0}.
\end{aligned} \tag{74}$$

$$\begin{aligned}
& \frac{d}{db'_{\bar{w}}}\left\{b'_{\underline{w}}\times Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)+b'_{\bar{w}}\times Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\right\} \\
= \;& \frac{\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)+\frac{\frac{\partial}{\partial b'_{\bar{w}}}\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}Q_{\bar{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right)\times b'_{\bar{w}}+b'_{\underline{w}}\times\frac{\partial}{\partial b'_{\bar{w}}}Q_{\underline{w}}\left(b'_{\underline{w}},b'_{\bar{w}}\right) \\
= \;& \frac{\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{R}\left(1-F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right)+\underbrace{\frac{\partial\Psi\left(b'_{\underline{w}},b'_{\bar{w}}\right)}{\partial b'_{\bar{w}}}}_{<0}\times\underbrace{\Delta\left(b'_{\underline{w}},b'_{\bar{w}}\right)}_{>0}.
\end{aligned}$$

The CEE conditional on the economy leaving the recession is, then:

$$\begin{aligned}
& \beta \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \tilde{b}}{Q\left(B\left(b'_{\bar{w}}, \bar{w}\right), \bar{w}\right) \times B\left(b'_{\bar{w}}, \bar{w}\right) + \bar{w} - b'_{\bar{w}}} \\
= \ & \frac{1}{R} + \underbrace{\frac{\frac{\partial}{\partial b'_{\bar{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}}_{<0} \left[\underbrace{\frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} - \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}}_{>0}\right] \\
< \ & \frac{1}{R}.
\end{aligned}$$

When $\beta R = 1$, this as the same as Equation (40).

The CEE conditional on a continuing recession is

$$\begin{aligned}
& \beta \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}} + Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}} + \underline{w} - \tilde{b}}{Q_{\underline{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\underline{w}}\left(b'_{\underline{w}}\right) + Q_{\bar{w}}\left(B_{\underline{w}}\left(b'_{\underline{w}}\right), B_{\bar{w}}\left(b'_{\underline{w}}\right)\right) \times B_{\bar{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b} \\
= \ & \frac{1}{R} + \underbrace{\frac{\frac{\partial}{\partial b'_{\underline{w}}} \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}{\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right)\left[1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right]}}_{>0} \left[\underbrace{\frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} - \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)}}_{>0}\right] \qquad (75) \\
> \ & \frac{1}{R},
\end{aligned}$$

When $\beta R = 1$, this is again the same as Equation (39).

(ii) Next, we show by a contradiction argument that, in equilibrium, $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. We proceed in two steps. First (step 1), we show that $b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. Next (step 2), we show that assuming $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$ would imply that $b'_{\bar{w}} > b'_{\underline{w}}$. However, by step 1, $b'_{\bar{w}} > b'_{\underline{w}} \Rightarrow \Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$, which establishes a contradiction.

Step 1: We prove that if $b'_{\bar{w}} > b'_{\underline{w}}$ then $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$.

First, define

$$\begin{aligned}
\theta_{\bar{w}}\left(b'_{\bar{w}}\right) &\equiv \frac{Q_{\bar{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\bar{w}}}{\Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} = \frac{1}{R}\left(\left(1 - F\left(\bar{\Phi}\left(b'_{\bar{w}}\right)\right)\right) b'_{\bar{w}} + \int_0^{\bar{\Phi}\left(b'_{\bar{w}}\right)} \left(\bar{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right)\right), \\
\theta_{\underline{w}}\left(b'_{\underline{w}}\right) &\equiv \frac{Q_{\underline{w}}\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \times b'_{\underline{w}}}{1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)} = \frac{1}{R}\left(\left(1 - F\left(\underline{\Phi}\left(b'_{\underline{w}}\right)\right)\right) b'_{\underline{w}} + \int_0^{\underline{\Phi}\left(b'_{\underline{w}}\right)} \left(\underline{\Phi}^{-1}(\phi) \times f(\phi)\ d\phi\right)\right),
\end{aligned}$$

where, recall, $\underline{\Phi}(x) > \bar{\Phi}(x)$ and $F\left(\underline{\Phi}(x)\right) > F\left(\bar{\Phi}(x)\right)$. Our goal is to show that if $b'_{\bar{w}} > b'_{\underline{w}}$ then $\theta_{\bar{w}}\left(b'_{\bar{w}}\right) > \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$. Suppose, first, that $b'_{\underline{w}} = b'_{\bar{w}} = x$. Then:

$$\begin{aligned}
\theta_{\bar{w}}(x) - \theta_{\underline{w}}(x) &= \underbrace{\frac{1}{R}\int_{\bar{\Phi}(x)}^{\underline{\Phi}(x)} \left(\left(x - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} \qquad (76) \\
&\quad + \underbrace{\frac{1}{R}\int_0^{\bar{\Phi}(x)} \left(\left(\bar{\Phi}^{-1}(\phi) - \underline{\Phi}^{-1}(\phi)\right) \times f(\phi)\ d\phi\right)}_{>0} > 0.
\end{aligned}$$

Since $\theta_{\bar{w}}\left(b'_{\bar{w}}\right)$ is increasing in $b'_{\bar{w}}$, and $\theta_{\underline{w}}\left(b'_{\underline{w}}\right)$ is increasing in $b'_{\underline{w}}$, equation (76) implies that for all $b'_{\bar{w}} > b'_{\underline{w}}$ $\theta_{\bar{w}}\left(b'_{\bar{w}}\right) > \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, implying $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. This concludes the proof of step 1.

Step 2: We prove that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0 \Rightarrow b'_{\bar{w}} > b'_{\underline{w}}$.

Suppose that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$. Then, the CEEs, (39)–(40), imply that

$$c'|_{H,\bar{w}} \leq c \leq c'|_{H,\underline{w}}. \tag{77}$$

Suppose, to derive a contradiction, that $b'_{\underline{w}} \geq b'_{\bar{w}}$. Recall that, if the economy leaves the recession, in equilibrium the country will keep its debt constant over time unless there is renegotiation, i.e., $b'' = B\left(b'_{\bar{w}}\right) = b'_{\bar{w}}$. Then,

$$\begin{aligned}
c'|_{H,\bar{w}} &= Q\left(b'_{\bar{w}}, \bar{w}\right) b'_{\bar{w}} + \bar{w} - b'_{\bar{w}} = \theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \bar{w} - b'_{\bar{w}} \\
&= \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \bar{w} - b'_{\bar{w}} \\
&\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\bar{w}}\left(b'_{\underline{w}}\right) + \bar{w} - b'_{\underline{w}} \\
&\geq \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\underline{w}}\left(b'_{\underline{w}}\right) + \bar{w} - b'_{\underline{w}} \\
&> \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\theta_{\bar{w}}\left(b'_{\bar{w}}\right) + \left(1 - \Psi\left(b'_{\underline{w}}, b'_{\bar{w}}\right)\right)\theta_{\underline{w}}\left(b'_{\underline{w}}\right) + \underline{w} - b'_{\underline{w}} = c'|_{H,\underline{w}}.
\end{aligned}$$

The first inequality follows from the assumption that $b'_{\underline{w}} \geq b'_{\bar{w}}$ and the fact that $(1-p)\,\theta_{\bar{w}}(x) - x < 0$ for any $p \in [0,1]$, which is due to $\theta_{\bar{w}}(x) \leq x/R < x$ for any x. The second inequality follows from the fact that $\theta_{\bar{w}}\left(b'_{\underline{w}}\right) \geq \theta_{\underline{w}}\left(b'_{\underline{w}}\right)$, see equation (76). The last inequality follows from the maintained assumption that $\bar{w} > \underline{w}$. We have therefore proven that if $b'_{\underline{w}} \geq b'_{\bar{w}}$ then $c'|_{H,\bar{w}} > c'|_{H,\underline{w}}$, which contradicts (77) and, hence, implies that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \geq 0$. We conclude that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) \leq 0$ implies that $b'_{\bar{w}} > b'_{\underline{w}}$.

Step 3: Step 1 and Step 2, taken together, imply that $\Delta\left(b'_{\underline{w}}, b'_{\bar{w}}\right) > 0$. ■

Proof of Proposition 8. The strategy of the proof is the same as that of Proposition 5. We prove the proposition by deriving a contradiction. To this aim, suppose that, for $\underline{\Pi}(b_w) = \underline{P}(\nu)$, the planner can deliver more utility to the agent than can the competitive equilibrium. Namely, $\nu > EV(b_w, \underline{w})$. Then, since $\underline{P}$ is a decreasing strictly concave function, we must have that $\underline{P}(EV(b_w, \underline{w})) > \underline{P}(\nu)$ and $\underline{P}'(EV(b, \underline{w})) > \underline{P}'(\nu)$. Note that, absent moral hazard, the price of recession-contingent debt is independent of the amount of recovery-contingent debt. It is therefore legitimate to define $\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) \equiv Q_{\underline{w}}\left(b_{\underline{w}}, b_{\bar{w}}\right)$.

First, the same argument invoked in the proof of Proposition 5 implies that $\underline{\Pi}(b_w) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}$. Hence,

$$\underline{P}\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) > \underline{P}(\nu) = \frac{R}{1-p}\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}. \tag{78}$$

where $EV\left(b_{\underline{w}}, \underline{w}\right)$ is decreasing in $b_{\underline{w}}$. Differentiating the two sides of the inequality (78) with respect to $b_{\underline{w}}$ yields:

$$\begin{aligned}
&\underline{P}'\left(EV\left(b_{\underline{w}}, \underline{w}\right)\right) \times \frac{d}{db_{\underline{w}}} EV\left(b_{\underline{w}}, \underline{w}\right) \\
> \;& \frac{R}{1-p}\frac{d}{db_{\underline{w}}}\left(\tilde{Q}_{\underline{w}}\left(b_{\underline{w}}\right) b_{\underline{w}}\right) = -\left[1 - F\left(\underline{\Phi}(b)\right)\right],
\end{aligned} \tag{79}$$

where the right-hand side equality follows from equation (74). Next, equation (48) implies that

$$\frac{d}{db}EV\left(b,\bar{w}\right)=-\left[1-F\left(\bar{\Phi}\left(b\right)\right)\right]\times u'\left(C^{H}\left(b,\bar{w}\right)\right),$$

where $C^{H}\left(b_{\underline{w}},\underline{w}\right)$ is the consumption level assuming that the recession-contingent debt $b_{\underline{w}}$ is honored. Plugging in the expression of $\frac{d}{db_{\underline{w}}}EV\left(b_{\underline{w}},\bar{w}\right)$ allows us to simplify (79) as follows:

$$u'\left(C^{H}\left(b_{\underline{w}},\underline{w}\right)\right)>-\frac{1}{\underline{P}'\left(EV\left(b_{\underline{w}},\underline{w}\right)\right)}. \tag{80}$$

Next, note that $C^{H}\left(b_{\underline{w}},\underline{w}\right)=c\left(\nu\right)$. Equation (80) yields $u'\left(c\left(\nu\right)\right)>-\frac{1}{\bar{P}'\left(EV\left(b_{\bar{w}},\bar{w}\right)\right)}$, while (71) yields that $u'\left(c\left(\nu\right)\right)=-\frac{1}{\underline{P}'\left(\nu\right)}$. Thus, the two conditions jointly imply that $-\frac{1}{\underline{P}'\left(\nu\right)}>-\frac{1}{\underline{P}'\left(EV\left(b_{\bar{w}},\underline{w}\right)\right)}$ which in turn implies that $\nu<EV\left(b_{\bar{w}},\underline{w}\right)$, since $\underline{P}$ is decreasing and concave. This contradicts the assumption that $\nu>EV\left(b_{\bar{w}},\underline{w}\right)$.

The analysis thus far establishes that $\nu\leq EV\left(b_{\bar{w}},\underline{w}\right)$. However, that $\nu<EV\left(b_{\bar{w}},\underline{w}\right)$ can also be safely ruled out, because it would contradict that the allocation chosen by the planner is constrained efficient. Therefore, $\nu=EV\left(b_{\bar{w}},\underline{w}\right)$. ■

Proof of Lemma 6 and Proposition 9. The Lagrangian of the planner's problem reads as

$$\begin{aligned}
\underline{\Lambda} \quad = \quad & \int_{\aleph}\left[\underline{w}-c_{\phi}+\beta\left(\left(1-p_{\phi}\right)\underline{P}\left(\underline{\omega}_{\phi}\right)+p_{\phi}\bar{P}\left(\bar{\omega}_{\phi}\right)\right)\right]dF\left(\phi\right)\\
& +\underline{\mu}\left(\int_{\aleph}\left(u\left(c_{\phi}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)\right)dF\left(\phi\right)-\nu\right)\\
& +\int_{\aleph}\underline{\lambda}_{\phi}\left(u\left(c_{\phi}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)-\underline{\nu}+\phi\right)d\phi\\
& +\int_{\aleph}\gamma_{\phi}\left(Z\left(b_{0}\right)-X\left(p_{\phi}\right)+\beta\left(\left(1-p_{\phi}\right)\underline{\omega}_{\phi}+p_{\phi}\bar{\omega}_{\phi}\right)\right)d\phi,
\end{aligned}$$

where the Lagrange multipliers of the PC and IC must be positive, $\underline{\lambda}_{\phi}\geq0,\gamma_{\phi}\geq0$. The first-order conditions yield:

$$f\left(\phi\right)=u'\left(c_{s}\right)\left(\underline{\mu}\,f\left(\phi\right)+\underline{\lambda}_{\phi}\right), \tag{81}$$

$$\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right)+\gamma_{\phi}=-\underline{P}'\left(\underline{\omega}_{\phi}\right)\,f\left(\phi\right), \tag{82}$$

$$\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right)+\gamma_{\phi}=-\bar{P}'\left(\bar{\omega}_{\phi}\right)\,f\left(\phi\right), \tag{83}$$

$$\beta\left(\bar{P}\left(\bar{\omega}_{\phi}\right)-\underline{P}\left(\underline{\omega}_{\phi}\right)\right)\,f\left(\phi\right)=\left(\underline{\lambda}_{\phi}+\underline{\mu}\,f\left(\phi\right)+\gamma_{\phi}\right)\left(X'\left(p_{\phi}\right)-\beta\left(\bar{\omega}_{\phi}-\underline{\omega}_{\phi}\right)\right), \tag{84}$$

while the envelope condition yields $\underline{P}'\left(\nu\right)=\underline{\mu}$.

The first order conditions (82)–(84) imply Equations (44)-(45) in the text. Since $\underline{P}$ and $\bar{P}$ are monotonic and concave, Equation (44) implies a positive relationship between $\underline{\omega}_{\phi}$ and $\bar{\omega}_{\phi}$. Equation (45) yields then a negative relationship between p_{ϕ} and $\underline{\omega}_{\phi}$. Consider, next, the IC constraint. When the IC constraint is binding, Equations (42), (44), and (45) pin down a unique solution for p_{ϕ}, $\underline{\omega}_{\phi}$ and $\bar{\omega}_{\phi}$, denoted by $\left(p^{*},\underline{\omega}^{*},\bar{\omega}^{*}\right)$. This establishes Lemma 6.

If $\nu>\underline{\omega}^{*}$ (case 1), the IC is not binding in the initial period. Moreover, by Proposition 6, promised utility is non-decreasing over time. Thus, the IC will never bind in the future, and can be ignored.

Suppose, next, that $\nu \leq \underline{\omega}^*$ (case 2). We first determine the upper bound realization of ϕ, denoted by ϕ^*, such that the PC is binding while the IC is not binding. Let $\left(c_\phi, p_\phi, \underline{\omega}_\phi, \bar{\omega}_\phi\right)$ denote the solution characterized in Proposition 6 when the IC is not binding and $(c^*_\phi, p^*, \underline{\omega}^*, \bar{\omega}^*)$ the solution characterized in Proposition 9 when the IC is binding. Note that c^*_ϕ is defined in (46). At the threshold realization ϕ^*, the two allocations must be equivalent, i.e.,

$$(c_{\phi^*}, p_{\phi^*}, \underline{\omega}_{\phi^*}, \bar{\omega}_{\phi^*}) = (c^*_{\phi^*}, p^*, \underline{\omega}^*, \bar{\omega}^*).$$

The promise-keeping constraint is satisfied irrespective of whether the IC is binding or not. Thus, the following relationship must hold true:

$$\begin{aligned}
\underline{\omega}^* &= \underline{\omega}_{\phi^*} = \int_0^{\phi^*} \left[u(c_\phi) - X(p_\phi) + \beta\left[(1-p_\phi)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right]\right] dF(\phi) \\
&\quad + \int_{\phi^*}^{\phi_{\max}} \left[u(c_{\phi^*}) - X(p_{\phi^*}) + \beta\left[(1-p_{\phi^*})\underline{\omega}_{\phi^*} + p_{\phi^*} \bar{\omega}_{\phi^*}\right]\right] dF(\phi) \\
&= \int_0^{\phi^*} (\underline{v} - \phi)\, dF(\phi) + (\underline{v} - \phi^*)\left[1 - F(\phi^*)\right] \\
&= \underline{v} - \int_0^{\phi^*} \phi dF(\phi) - \phi^*\left[1 - F(\phi^*)\right]. \qquad (85)
\end{aligned}$$

Since $\underline{\omega}_\phi$ is decreasing in ϕ, then ϕ^* is unique. Moreover, if $\phi < \phi^*$, then $\underline{\omega}_\phi > \underline{\omega}^*$. In this case, the solution is not history-dependent and is determined as in Proposition 6 (case 2.a). If, to the opposite, $\phi \geq \phi^*$, then $\underline{\omega}_\phi = \underline{\omega}^*$. Two subcases must be distinguished here. First, if $\phi \geq \phi^*$ and $\nu = \underline{\omega}^*$, then the multipliers of both the IC and PC must equal, $\underline{\lambda}_\phi = \gamma_\phi = 0$, because the planner keeps the triplet $(p^*, \underline{\omega}^*, \bar{\omega}^*)$ constant. More formally, the envelope condition together with Equation (82) implies that

$$\underline{P}'(\nu) = \underline{P}'\left(\underline{\omega}_\phi\right) + \frac{\underline{\lambda}_\phi + \gamma_\phi}{f(\phi)}.$$

Thus, $\nu = \underline{\omega}_\phi = \underline{\omega}^*$, and both the multiplier of the IC and that of the PC must be zero. In other words, as long as the IC was binding in the previous period, and keeps binding in the current period, the planner keeps consumption, effort and promised utilities constant.

Second, if $\phi \geq \phi^*$ and $\nu < \underline{\omega}^*$, then the planner must adjust promised utility, $\underline{\omega}_\phi = \underline{\omega}^*$, to satisfy the IC. In this case, the multiplier of the IC must be strictly positive, $\gamma_\phi > 0$.

For the determination of consumption, two separate cases must be distinguished. In the first case (2.b), ϕ is not very large, and both the IC and the PC bind. In this case, $\underline{\lambda}_\phi > 0$, and the IC and the PC determine jointly the consumption level, whose level is given by c^*_ϕ as defined in Equation (46). In the second case (2.c), ϕ is sufficiently large, and the PC does not bind. In this region, $\underline{\lambda}_\phi = 0$, and the planner provides a consumption level that is consistent with the promise-keeping constraint, which is pinned down by Equation (47).

Next, we determine the unique threshold, $\tilde{\phi}(\nu)$, that sets apart case (2.b) from case (2.c). More precisely, if $\phi \leq \tilde{\phi}(\nu)$, then both the IC and the PC bind and the characterization of consumption in Equation (46) applies. Conversely, if $\phi > \tilde{\phi}(\nu)$, then only the IC binds and the characterization of consumption in Equation (47) applies. Because the PC holds with equality at the threshold realization $\tilde{\phi}(\nu)$, the consumption level for all realizations of $\phi > \tilde{\phi}(\nu)$ where the PC is not binding must be given by

$$c^*_{\tilde{\phi}(\nu)} = u^{-1}\left(\underline{\nu} - \tilde{\phi}(\nu) - Z\left(b_0\right)\right).$$

This condition, together with the promise-keeping constraint, fully characterizes the threshold $\tilde{\phi}(\nu)$:

$$
\begin{aligned}
\nu &= \int_0^{\phi^*} \left[u(c_\phi) - X(p_\phi) + \beta\left[(1-p_\phi)\underline{\omega}_\phi + p_\phi \bar{\omega}_\phi\right]\right] dF(\phi) \\
&\quad + \int_{\phi^*}^{\tilde{\phi}(\nu)} \left[u(c^*_\phi) - X(p^*) + \beta\left[(1-p^*)\underline{\omega}^* + p^* \bar{\omega}^*\right]\right] dF(\phi) \\
&\quad + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[u(c^*_{\tilde{\phi}(\nu)}) - X(p_{\phi^*}) + \beta\left[(1-p_{\phi^*})\underline{\omega}_{\phi^*} + p_{\phi^*} \bar{\omega}_{\phi^*}\right]\right] dF(\phi) \\
&= \int_0^{\phi^*} (\underline{v} - \phi)\, dF(\phi) + \int_{\phi^*}^{\tilde{\phi}(\nu)} (\underline{v} - \phi)\, dF(\phi) + \int_{\tilde{\phi}(\nu)}^{\phi_{\max}} \left[u(c^*_{\tilde{\phi}(\nu)}) + Z(b_0)\right] dF(\phi) \\
&= F(\tilde{\phi}(\nu))\underline{v} - \int_0^{\tilde{\phi}(\nu)} \phi dF(\phi) + \left(\underline{v} - \tilde{\phi}(\nu)\right)\left[1 - F(\tilde{\phi}(\nu))\right] \\
\nu &= \underline{v} - \int_0^{\tilde{\phi}(\nu)} \phi dF(\phi) - \tilde{\phi}(\nu)\left[1 - F(\tilde{\phi}(\nu))\right]. \qquad (86)
\end{aligned}
$$

Finally, we prove that $\tilde{\phi}(\nu) \geq \phi^*$. More precisely, if $\nu = \underline{\omega}^*$ then Equations (85) and (86) imply that $\phi^* = \tilde{\phi}(\nu)$. In this case, consumption remains constant at the level $c^*_{\phi^*}$ as long as the IC is binding (i.e., if and only if $\phi > \phi^*$). However, if $\nu < \underline{\omega}^*$, then $\tilde{\phi}(\nu) \geq \phi^*$. Namely, there is a positive range of high realizations of ϕ such that the IC is binding while the PC is not binding. In this range, consumption will be lower in the initial period, i.e., it is pinned down by Equation (47). As of the second period, $c^*_{\phi^*}$ as given as in Equation (47) provides a lower bound to consumption. This concludes the proof. ■