A Theory of Voter Turnout

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Abstract. I consider a two-candidate election in which there is aggregate uncertainty over the popularities of the candidates, where voting is costly, and where voters are instrumentally motivated. The unique equilibrium predicts substantial turnout under reasonable conditions, and greater turnout for the apparent underdog helps to offset the expected advantage of the perceived leader. I also present predictions about the response of turnout and the election outcome to various parameters, including the importance of the election; the cost of voting; the perceived popularity of each candidate; the relative preference intensities of different partisans; the positions of candidates on an underlying policy spectrum; and the accuracy of pre-election information sources, such as opinion polls. Amongst other results, I show that a candidate enjoys greater electoral success by having a smaller group of fanatical supporters rather than a larger group of backers with milder preferences, I demonstrate that in a benchmark case the election outcome is not influenced by candidates' positions on a left-right policy spectrum, and I evaluate whether a voluntary-turnout system picks the right winner.

Why do people vote? Across different types of voters, how is turnout likely to vary? Will the result reflect accurately the pattern of preferences throughout the electorate? Does a system with voluntary turnout select the right candidate? How do the policy positions of competing candidates influence their turnout rates, their electoral prospects, and so (ultimately) their manifesto commitments? These questions are central to the study of democratic systems. Nevertheless, the leading turnout question ("why do people vote?") is problematic: the classic paradox of voting alleges that a costly vote's influence is too small to justify the participation of an instrumentally motivated voter. Moreover, the lack of an accepted canonical model of voter turnout frustrates the answers to the other questions posed.

Here I argue that a theory of turnout based upon instrumentally motivated actors works well. I model a two-candidate election where voting is costly, where voters are instrumentally motivated, and where the substantive and reasonable departure from most established theories is this: there is aggregate uncertainty about the popularities of the candidates.

Acknowledgements. This paper supersedes an earlier working paper (Myatt, 2012a): it corrects an omission from the proof of Lemma 1, it integrates Myatt (2012b), and it reports new welfare results. The broader project, associated presentations, and conference drafts date back fifteen years, and so over this very long period I have spoken with very many colleagues about it. Given this length of time, I apologize to anyone whom I have neglected to mention. Particular thanks go to Jean-Pierre Benoît, Micael Castanheira, Torun Dewan, Marco Faravelli, Steve Fisher, Libby Hunt, Clare Leaver, Joey McMurray, Adam Meirowitz, Tiago Mendes, Becky Morton, Debraj Ray, Kevin Roberts, Norman Schofield, Ken Shepsle, Chris Wallace, and Peyton Young.

I find that the unique equilibrium is consistent with substantial voter turnout under reasonable conditions. I evaluate the response of turnout rates to the importance of the election, to the cost of voting, to the perceived popularities of the candidates, to voters' preference intensities, to candidates' policy positions, and to voters' pre-election information. Amongst other implications, I show that a candidate can achieve greater electoral success with a smaller group of fanatical supporters rather than with a larger group of backers with milder preferences, I find that in a benchmark case the election outcome is not influenced by candidates' positions on a policy spectrum, and I evaluate whether the election picks the right winner. These results demonstrate that the model can be a useful component of broader policy-and-voting models with endogenous voter participation.

To illustrate the leading claim (that substantial turnout is predicted by a reasonable rational-choice model) consider an electoral region with 100,000 citizens (a small city or a parliamentary constituency) of whom 75,000 (a typical suffrage rate of 75%) are eligible to vote. Voters' beliefs are summarized by a 95% confidence interval for the popularity of the leading candidate (the proportion of who prefer her) which ranges from 57% to 62%. (This would emerge from a pre-election opinion poll with a typical sample size.) Finally, each voter is willing to participate for a 1-in-2,500 chance of changing the outcome. This scenario maps into a set of parameter choices for the model. Under these parameters, the (unique) equilibrium predicts an overall turnout rate of over 50%, with greater turnout for the expected underdog offsetting her popularity disadvantage. This is a special case of the approximate solution:

$$\text{turnout rate} \approx \frac{\text{instrumental benefit / voting cost}}{\text{population} \times \text{width of 95\% confidence interval}}, \tag{$\star$}$$

A key factor is a voter's willingness to participate. In the illustration, he "is willing to participate for a 1-in-2,500 chance" so that his benefit from changing the electoral outcome for 100,000 people is 2,500 times the cost of voting; this is the numerator of ($\star$). In the context of a city-wide election it seems that the required probability is not outrageously small. Moreover, if the cost of voting is low then reasonable turnout levels can even be explained by narrow private concerns over the election outcome.

For larger elections, the rule-of-thumb ($\star$) implies that voters need to show up for a 1-in-125,000 influence to generate 50% turnout from a population of 5,000,000. This approximates an election such as the recent Scottish independence referendum. In September 2014 those eligible to vote (an electorate of 4.3 million out of a population of 5.3 million) were asked:[2] "Should Scotland be an independent country?" The required odds of influence might be seen as too small for someone who is concerned only with the narrow short-term private consequences of the election. In this Scottish case, however, the odds do not seem too outrageous given what was at stake. Indeed, turnout was high: nearly 85%.

In more mundane elections, very mild social preferences (that is, a concern with the impact of the election on others) can also make things work. To illustrate this, consider a voter who believes (paternalistically, perhaps) that a win by his preferred candidate will improve the

[2]The vote was granted to those aged 16 or older (reduced from the usual 18 years in the United Kingdom) which (together with a high voter registration rate) yielded a relatively large electorate.

life of every citizen by $250 per annum over a five-year term. Suppose that his personal voting cost is $5. If his concern for others is only 0.01% (that is, 1-in-10,000) then, in a population of 5,000,000, he will be willing to participate at 1-in-125,000 odds.

The confidence interval that forms part of the rule-of-thumb ($\star$) arises from uncertainty over the popularities of the candidates. For most voting models in the literature, each voter's type (and so his voting decision) is an independent draw from a known distribution. The independence assumption implies that, in a large electorate, voters are able to predict almost perfectly the election outcome. Except in knife-edge cases, it also implies that the probability of a tie (and so the incentive to vote) declines exponentially with the electorate size. If, as in this paper, aggregate uncertainty is present (so that, more reasonably, voters are unsure which candidate is more popular) then the tie probability is inversely proportional to the electorate size. This higher probability translates into higher turnout.

Turnout is also bolstered by an underdog effect. A supporter of the underdog (the candidate expected to be less popular) updates his beliefs based upon his own type realization, and so is less pessimistic about the underdog's position than a supporter of her opponent. He sees a tie as more likely, and this induces (in equilibrium) greater turnout for the underdog. If all voters share the same cost of voting and the same benefit from their favored election outcome, then this underdog effect exactly offsets any expected popularity advantage for a candidate. More generally, the underdog effect pushes the election toward a closer race. This raises the likelihood of a tie, and so generates higher turnout.

I also offer comparative-static predictions. If voters differ only by the identity of their preferred candidate, then the rule-of-thumb ($\star$) operates: turnout increases with the importance of the election and with the precision of voters' beliefs about the candidates' popularities, but decreases with the cost of voting. If voting costs differ (either within or between the factions of voters) then turnout reacts non-monotonically to voters' information.

The results have implications for the success and failure of the candidates. The underdog effect means that an increase in the popularity of a candidate is unwound by reduced relative turnout for her. However, changes in the intensity of voters' preferences do matter. If a candidate's supporters care more about the election outcome than the supporters of her opponent, then she benefits from increased relative turnout. Thus a candidate for office does better by having a smaller faction of fanatical supporters than a larger faction of moderate supporters. What really matters for electoral success is the strength of feeling amongst one faction's members relative to another. These strengths of feeling are influenced, in turn, by the candidates' policy positions.

If candidates for office recognize that the election outcome is determined by the turnout rates of their supporters then their policy positions may differ from those in a world with compulsory turnout. Consider, for example, a shift away from the center of the policy space. This might appeal to the candidate's supporters, so strengthening their feelings and raising turnout amongst this faction. However, such a move antagonizes the opposing faction (they dislike more strongly the extreme position) and so strengthens support (and turnout) for the opposing candidate. Hence, a shift outward raises turnout rates on both sides. If strengths of feeling in the two factions rise proportionally (this is true when preferences are linear in the difference between a candidate's policy and a faction's ideal point) then there is no change in the ratio of turnout rates. What this means is that there are circumstances in which policy shifts do not influence election outcomes. More generally, the net gain from a policy shift depends on the size of policy concerns for each faction relative to any valence (quality) advantage (or disadvantage) for that faction's candidate. A high-valence candidate from a faction with relatively weak policy concerns gains by a move toward the center, while her opponent does better by shifting her policy platform outward.

Beyond comparative-static results, I also evaluate the social performance of voting. The underdog effect means that the election is biased against the perceived leader. This means that the election selects the right winner only when the candidates' popularities are perceived equally. Turnout does, however, reflect appropriately different strengths of feelings amongst competing factions. (Under compulsory voting, strengths of feeling do not influence the outcome.) More generally, an election with voluntary turnout performs better than simply picking the perceived leader when the candidates are sufficiently close and when the prior beliefs about their popularities are not too precise. Contrary to claims within the recent literature, I find reasonable conditions under which the election is biased against the utilitarian choice whenever she enjoys an advantage in expected popularity.

This paper joins a resurgent interest in elections with endogenous turnout (Krishna and Morgan, 2011, 2012, 2014, 2015; Evren, 2012; Faravelli and Sanchez-Pages, 2012; Faravelli, Man, and Walsh, 2013; Herrera, Morelli, and Palfrey, 2014; Kartal, 2015). Most papers, as well as the classic literature, specify models without aggregate uncertainty. Elsewhere I have used aggregate uncertainty to model strategic voting (Myatt, 2007; Dewan and Myatt, 2007), protest voting (Myatt, 2015), and district competition (Myatt and Smith, 2014). One contribution here is to show how aggregate uncertainty can be used to resolve the traditional turnout paradox. Notably, however, Evren (2012) independently described a related turnout model with aggregate uncertainty over a fraction of voters are who altruistic (or ethical), and recent work-in-progress by Krishna and Morgan (2015) is incorporating the aggregate uncertainty from an ancestor of this paper (Myatt, 2012a). Relative to Evren (2012), this paper considers a finite-population model with an explicit solution, a full suite of comparative-static results, an analysis of candidate policy choices, and an evaluation of the social performance of voluntary voting. In contrast to Krishna and Morgan (2015), this paper shows that voluntary voting does not typically select the utilitarian winner.

In the next section I describe a model of voluntary and costly voting in a two-candidate election and characterize optimal turnout behavior (§1). I pause to study the properties of beliefs in elections with aggregate uncertainty (§2), before characterizing the unique equilibrium (§3) and its basic comparative-static properties (§4). I extend the model to consider asymmetric and idiosyncratic voting costs (§5), to ask whether the election picks the right winner (§6), to characterize the policy positions of office-seeking candidates (§7), and to incorporate other-regarding preferences in larger electorates (§8). After surveying some related literature (§9), I conclude with some take-home messages regarding the turnout paradox (§10).

1. A Model of a Plurality Rule Election

The Election. Within an electorate comprising $n+1$ voters, each member ("he") votes either for candidate L (left, "she") or for candidate R (right, also "she"). The candidate with the most votes wins. A fair coin breaks any tie. The results also hold for other tie-break rules.

A randomly chosen voter prefers R with probability p and L with probability $1-p$, and so p is the true popularity of R relative to L. Conditional on p, types are independent. However, there is aggregate uncertainty: p is drawn from a bounded density $f(\cdot)$ with mean $\bar{p}$. Each voter updates this common prior based on his own type realization.

I also allow for aggregate uncertainty about the precise electorate size, although this turns out to have little importance. A voter is available to vote with independent probability a, where a is drawn from the density $g(\cdot)$ with mean $\bar{a}$. Hence, if everyone who is able to do so turns out to vote then the expected turnout is $\bar{a}(n+1)$. The model also extends straightforwardly to allow for an (uncertain) probability that some voters always vote.

For convenience I make the technical assumptions that the densities $f(\cdot)$ and $g(\cdot)$ are both continuous with bounded first derivatives and with full support on $[0,1]$.

Voting is voluntary, but costly: a voter incurs a cost $c > 0$ if he goes to the polls. A voter enjoys a benefit $u > 0$ if and only if his preferred candidate wins. I assume that $u > 2c$ so that some turnout is possible. For now, u and c are common to everyone. However, Section 5 allows for asymmetric and heterogeneous costs and benefits.

The key decision for a voter is one of participation. (If he turns out then he optimally votes for his favorite.) I examine type-symmetric strategies in which voters of the same type (L or R) behave in the same way. Mostly (but not always) I focus on "incomplete turnout" situations in which not everyone shows up. A strategy profile that fits these criteria reduces to a pair of probabilities $t_R \in (0,1)$ and $t_L \in (0,1)$. These turnout rates amongst the two electoral factions generate an overall expected turnout rate of $\bar{t} = \bar{a}(\bar{p}t_R + (1-\bar{p})t_L)$.

Optimal Voting. Here I consider the decision faced by a voter as he considers the likely outcome amongst the other n members of the electorate. I write b_L and b_R for the vote totals amongst those other electors. The number of abstentions is $n - b_L - b_R$.

Consider a supporter of candidate R. If there is a tie amongst others (that is, if $b_R = b_L$) and if the tie break goes against R, then his participation is pivotal to a win for R rather than L. Similar, if there is a near-tie, by which I mean that $b_R = b_L - 1$, and the tie break is favorable, then his participation is again pivotal. In other circumstances he has no influence. Assembling these observations, and using similar reasoning for a supporter of L,

$$\Pr[\text{pivotal} \mid R] = \frac{\Pr[b_R = b_L \mid R] + \Pr[b_R = b_L - 1 \mid R]}{2}, \quad \text{and} \tag{1}$$

$$\Pr[\text{pivotal} \mid L] = \frac{\Pr[b_R = b_L \mid L] + \Pr[b_L = b_R - 1 \mid L]}{2}. \tag{2}$$

A supporter of R finds it strictly optimal to participate if and only if the expected benefit from voting exceeds the cost, so that $u\Pr[\text{pivotal} \mid R] > c$. If this inequality (and the inequality for a supporter of L) holds then, given that c and u are common to everyone, turnout will be complete. However, as turnout increases (that is, as the turnout probabilities t_R and t_L rise) the pair of pivotal probabilities typically fall. If the turnout strategies ensure that the expected costs and benefits of voting are equalized for both voter types then these strategies yield an equilibrium. (Formally: a type-symmetric Bayesian Nash equilibrium in mixed strategies.) For parameters in an appropriate range, such an incomplete-turnout equilibrium (where $1 > t_R > 0$ and $1 > t_L > 0$) is characterized by a pair of equalities:

$$\Pr[\text{pivotal} \mid R] = \Pr[\text{pivotal} \mid L] = \frac{c}{v}. \tag{3}$$

Conceptually, an equilibrium characterization is straightforward: I must find a pair t_L and t_R such that these two equalities are satisfied. However, in general the pivotal probabilities depend on t_L, t_R, $f(\cdot)$, $g(\cdot)$, and n in a complex way. These probabilities are rather more tractable in larger electorates. I show this in the next section.

2. Election Outcomes with Aggregate Uncertainty

Votes are often modeled (in the literature) as independent draws from a known distribution. However, if type probabilities are unknown then votes are only conditionally independent. Unconditionally there is correlation between the ballots.[3] Here I consider the properties of beliefs about various electoral events of interest in the presence of aggregate uncertainty.

Outcome Probabilities in Large Electorates. Taking a (temporary) step outside the two-candidate model, consider an election with $m + 1$ options, so that there are m candidates plus abstention. Suppose that the electoral situation is described by $v \in \Delta$ where Δ is the m-dimensional unit simplex. v is a vector of voting probabilities: a randomly selected elector votes for candidate i with probability v_i, and abstains with probability $v_0 = 1 - \sum_{i=1}^{m} v_i$. While v can be interpreted as the underlying electoral support for the different candidates, it does not necessarily represent their true popularities. The distinction is because v_i is the probability that an actually elector votes for i, and not the probability that he prefers her. So, adapting the notation of the two-candidate model, if the supporters of candidate i turnout with probability t_i then $v_i = ap_it_i$.

Even if v is known (with aggregate uncertainty, it is not) then the election outcome remains uncertain owing to the idiosyncrasies of individual vote realizations. That outcome is represented by $b \in \Delta_n^\dagger$, where $\Delta_n^\dagger \equiv \left\{b \in \mathcal{Z}_+^{m+1} \mid \sum_{i=0}^{n} b_i = n\right\}$ and b_i is the number of votes cast for candidate i. Conditional on v, the outcome b is a multinomial random variable. However, suppose that the underlying support for the options (the m candidates and abstention)

[3]It is natural to think of voters as symmetric. However, independently drawing their types from the same distribution is an excessively strong form of symmetry. A weaker form of symmetry is that beliefs do not depend on the labeling of voters, so that they are exchangeable in the sense of de Finetti (see, for instance, Hewitt and Savage, 1955). Indeed, if a potentially infinite sequence of voters can be envisaged then (at least for binary realizations L and R) exchangeability ensures a conditionally independent representation.

is unknown. Specifically, suppose that beliefs about v are represented by a continuous and bounded density $h(\cdot)$ ranging over Δ. Taking expectations over v,

$$\Pr[b \,|\, h(\cdot)] = \int_{\Delta} \frac{\Gamma(n+1)}{\prod_{i=0}^{m} \Gamma(b_i+1)} \left[\prod\nolimits_{i=0}^{m} v_i^{b_i}\right] h(v)\, dv, \tag{4}$$

where the Gamma function $\Gamma(\cdot)$ satisfies $\Gamma(x+1) = x!$ for $x \in \mathcal{N}$. This expression is complex, but for larger n what matters is the density $h(\cdot)$ evaluated at the peak of $v_i^{b_i}$. That peak occurs at $v = \frac{b}{n}$. Note that $\prod_{i=0}^{m} v_i^{b_i}$ is sharply peaked around its maximum, and increasingly so as n grows. For large n only the density $h\left(\frac{b}{n}\right)$ really matters, and so

$$\Pr[b \,|\, h(\cdot)] \approx h\left(\frac{b}{n}\right) \int_{\Delta} \frac{\Gamma(n+1)}{\prod_{i=0}^{m} \Gamma(b_i+1)} \left[\prod\nolimits_{i=0}^{m} v_i^{b_i}\right] dv = \frac{\Gamma(n+1)}{\Gamma(n+m+1)} h\left(\frac{b}{n}\right), \tag{5}$$

where the final equality follows because the integrand is a (scaled) Dirichlet density.

This kind of logic was used by Good and Mayer (1975) and by Chamberlain and Rothschild (1981). They considered ties in two-option electins (the event $b_0 = b_1 = \frac{n}{2}$ for $m = 1$ where n is even) and demonstrated that $n \Pr[\text{tie}] \to h\left(\frac{1}{2}, \frac{1}{2}\right)$. The same logic generalizes to larger m and to more general electoral outcomes. This is confirmed by Lemma 1.[4]

Lemma 1 (Outcome Probabilities in Large Electorates)**.** *If voting probabilities are described by a continuous density* $h(\cdot)$ *with bounded derivatives, then* $\lim_{n\to\infty} \max_{b \in \Delta_n^\dagger} |n^m \Pr[b] - h(b/n)| = 0$.

An implication is that what matters when thinking about large elections is not idiosyncratic type realizations but rather the density $h(\cdot)$ that describes aggregate uncertainty. The law of large numbers ensures that any idiosyncratic noise is averaged out. In a committee with few voters idiosyncratic noise remains and so a model which specifies only idiosyncratic uncertainty can be useful. When there are more voters, however, "independent type" models are discomforting: the modeler is forcing beliefs to be entirely driven by factors (idiosyncratic type realizations) which are eliminated when there is aggregate uncertainty.

Pivotal Probabilities. I now reconsider pivotal events in two-candidate elections. I drop the m-candidate notation and return to the (L, R)-notation used throughout the remainder of the paper. For a voter with type $i \in \{L, R\}$ who holds beliefs $h(v \,|\, i)$ about the voting probabilities of others, the probability of an exact tie (a near tie is similar) is

$$\begin{aligned}\Pr[b_L = b_R \,|\, i] &= \sum\nolimits_{z=0}^{\lfloor n/2 \rfloor} \Pr[b_L = b_R = z \,|\, i] \\ &= \sum\nolimits_{z=0}^{\lfloor n/2 \rfloor} \frac{\Gamma(n+1)}{[\Gamma(z+1)]^2 \Gamma(n-2z+1)} \int_{\Delta} (v_L v_R)^z v_0^{n-2z} h(v \,|\, i)\, dv,\end{aligned} \tag{6}$$

where $v \in \Delta$ combines the voting probabilities v_L, v_R, and $v_0 = 1 - v_L - v_R$, and where $\lfloor n/2 \rfloor$ is the integer part of $\frac{n}{2}$. If n is (moderately) large then Lemma 1 can be exploited and the

[4]Good and Mayer (1975) and Chamberlain and Rothschild (1981) considered elections with two options (so $m = 1$ in the notation of this section) where n is even. They considered the probability of a tied outcome. This corresponds to a sequence of election outcomes of the form $b^n = \left(\frac{n}{2}, \frac{n}{2}\right)$, which obviously satisfies $\lim_{n\to\infty} \left(\frac{b^n}{n}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Equation (2) from Good and Mayer (1975) corresponds to Lemma 1 for this special case; Proposition 1 from Chamberlain and Rothschild (1981) reports a rediscovery of the same result.

probability in (6) can be approximated with a simpler expression. Using Lemma 1,

$$\Pr[b_L = b_R \,|\, i] \approx \frac{1}{n}\sum_{z=0}^{\lfloor n/2 \rfloor} \frac{h\big(1-\frac{2z}{n}, \frac{z}{n}, \frac{z}{n} \,|\, i\big)}{n}. \tag{7}$$

Allowing n to grow, the summation defines a Riemann integral of $h(1-2x, x, x \,|\, i)$ over the range $x \in [0, 1/2]$. Dealing with these heuristic steps more carefully yields another lemma.

Lemma 2 (Pivotal Probabilities in Large Electorates)**.** *If beliefs about the probabilities of abstention and votes are described by the density $h(v_0, v_L, v_R \,|\, i)$ then the probabilities of a tie and a near tie are asymptotically equivalent:* $\lim_{n\to\infty} n \Pr[b_L = b_R \pm 1 \,|\, i] = \lim_{n\to\infty} n \Pr[b_L = b_R \,|\, i]$*. Moreover,*

$$\lim_{n\to\infty} n \Pr[\text{pivotal} \,|\, i] = \int_0^{1/2} h(1-2x, x, x \,|\, i)\, dx \quad \textit{where} \quad i \in \{L, R\}. \tag{8}$$

Notice that (in the limit) the probabilities of tie and near-tie events are the same. If there were no aggregate uncertainty, then these probabilities would be very different. This is easy to see when n is odd and there is no abstention, so that $v_0 = 0$. If v is known, then

$$\Pr[b_L = b_R \pm 1] = \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+1}{2}+1\right)\Gamma\left(\frac{n-1}{2}+1\right)} v_L^{(n\pm1)/2} v_R^{(n\mp1)/2} \quad\Rightarrow\quad \frac{\Pr[b_L = b_R + 1]}{\Pr[b_L = b_R - 1]} = \frac{v_L}{v_R} \neq 1, \tag{9}$$

where "$\neq 1$" holds if and only if $v_L \neq v_R$. Equation (9) illustrates a worrying property of IID models; surely two close events should not have radically different probabilities?[5]

From Lemma 2, a voter's beliefs about pivotal events are determined by the density $h(\cdot \,|\, i)$ over voting probabilities. This, density emerges from his beliefs about the availability and preferences of others and from their turnout rates. Each elector is available with probability a and prefers R to L with probability p, and so the underlying electoral situation is described by $(a, p) \in [0, 1]^2$ with density $f(p)g(a)$. However, a voter updates his beliefs based on his own availability and his own preference, and so I write $f(p \,|\, i)$ where $i \in \{L, R\}$ and $g(a \,|\, \text{available})$ for these posterior beliefs. Beliefs about a and p must be transformed into beliefs about v_L, v_R, and the abstention probability $v_0 = 1 - v_L - v_R$. Turnout rates of t_R and t_L yield $v_R = apt_R$ and $v_L = a(1-p)t_L$. The Jacobian is readily obtained:

$$\frac{\partial(v_R, v_L)}{\partial(p, a)} = \begin{bmatrix} at_R & pt_R \\ -at_L & (1-p)t_L \end{bmatrix} \quad\Rightarrow\quad \left|\frac{\partial(v_L, v_R)}{\partial(p, a)}\right| = at_Lt_R. \tag{10}$$

Looking back to Lemma 2, the density $h(\cdot \,|\, i)$ is only evaluated where $v_L = v_R = x$. Using these inequalities it is straightforward to solve for p and a, and so

$$h(1-2x, x, x \,|\, i) = \frac{f(p^\star \,|\, i)g(a \,|\, \text{available})}{at_Lt_R} \text{ where } p^\star \equiv \frac{t_L}{t_L + t_R} \text{ and } a = \frac{x(t_R + t_L)}{t_Rt_L}. \tag{11}$$

Here $p^\star$ is a critical threshold for R's popularity relative to L. It is the underlying popularity that R needs to enjoy if she is to offset any difference in turnout rates; if (and only if) $p > p^\star$

[5]Some researchers have relied on the property reported in equation (9). For example, Taylor and Yildirim (2010a) employed an IID specification, and in their model an equilibrium requires different types to perceive the same probability of pivotality, and so $\Pr[b_L = b_R + 1] = \Pr[b_L = b_R - 1]$. Given (9), this can only be true if $v_L = v_R$, and so turnout must be inversely proportional to the popularity of a candidate, so that $(1-p)t_L = pt_R$. Once aggregate uncertainty is introduced, their argument no longer applies. Fortunately, however, their conclusion (turnout should be inversely related to perceived popularity) remains, as I confirm in this paper.

then R is more likely to win than L (and such a win becomes very likely in a large electorate). For instance, if $t_L = t_R = \frac{1}{2}$ then $p^\star = \frac{1}{2}$, and so R needs only to be the most popular to win; however, if $t_L = 2t_R$, so that L's supporters are twice as likely to turn up to the polling booth, then $p^\star = \frac{2}{3}$, and R needs to enjoy much greater popularity if she is to beat her opponent.

Looking again to Lemma 2, the density $h(1-2x, x, x \mid i)$ integrates to yield

$$\lim_{n\to\infty} n \Pr[\text{pivotal} \mid i] = \int_0^{1/2} h(1-2x,x,x \mid i)\,dx = \frac{f(p^\star \mid i)}{t_L + t_R} \int_0^1 \frac{g(a \mid \text{available})}{a}\,da. \tag{12}$$

A tied outcome is only really feasible when p is close to $p^\star$, and so when contemplating the likelihood of a pivotal event a voter asks how likely this is by evaluating the density $f(p^\star \mid i)$.

Equation (12) relies on the conditional beliefs about p and a. Using Bayes' rule,

$$g(a \mid \text{available}) = \frac{g(a)a}{\bar{a}}, \quad f(p \mid L) = \frac{f(p)(1-p)}{1-\bar{p}} \quad \text{and} \quad f(p \mid R) = \frac{f(p)p}{\bar{p}}, \tag{13}$$

where $\bar{a}$ is the prior expected availability of voters, and $\bar{p}$ is the prior expected popularity of R relative to L. Using these updated beliefs generates the following result.

Lemma 3 (Conditional Pivotal Probabilities in Large Electorates)**.** *If the supporters of L and R participate with probabilities t_L and t_R then, from the perspective of the $(n+1)$st voter,*

$$\lim_{n\to\infty} n \Pr[\text{pivotal} \mid L] = \frac{f(p^\star)}{\bar{a}(t_L+t_R)} \frac{1-p^\star}{1-\bar{p}} \quad \textit{and} \tag{14}$$

$$\lim_{n\to\infty} n \Pr[\text{pivotal} \mid R] = \frac{f(p^\star)}{\bar{a}(t_L+t_R)} \frac{p^\star}{\bar{p}}, \quad \textit{where} \quad p^\star = \frac{t_L}{t_L+t_R}, \tag{15}$$

and where $\bar{p}$ is the expected popularity of R and $\bar{a}$ is the expected availability of voters.

This lemma recycles the notation "$p^\star$" for the critical threshold of R's popularity relative to L. $v_L = v_R$ if and only if $p = p^\star$, and so if R is to win then her popularity must exceed $p^\star$.

Several other aspects of Lemma 3 are worthy of note. Firstly, the likelihood of a pivotal outcome is, of course, inversely proportional to the electorate size n.[6] A consequence is that the relative size of benefits and costs, captured by $\frac{v}{c}$, needs to be larger in a larger electorate if the same turnout rates are to be supported. Secondly, and relatedly, the pivotal probability is inversely proportional to the turnout rates and to the expected availability of voters. Thirdly, the expression in (12) suggested that the probability of a tie is more likely when a is uncertain; this is because $1/a$ is a convex function, and so $\int_0^1 \frac{g(a \mid \text{available})}{a}\,da$ increases if $g(\cdot \mid \text{available})$ becomes riskier in the usual sense. However, once updated beliefs are considered the riskiness of $g(\cdot)$ is unimportant, and so uncertainty of the electorate size plays no real role. Finally, and perhaps most interestingly, the probability of a tied outcome depends on the nature of the ex ante beliefs $f(p)$ about the relative support of the candidates. The probability is higher as $p^\star = t_L/(t_L+t_R)$ moves closer to the mode of $f(\cdot)$.

[6] As observed by Good and Mayer (1975), this is not the case when votes are independent draws. Under an IID specification, the probability of a pivotal event is inversely proportional to the square root of the electorate size in the knife-edge case where the underlying support of the candidates is balanced; otherwise, the probability disappears exponentially with the electorate size (Beck, 1975; Margolis, 1977; Owen and Grofman, 1984).

3. EQUILIBRIUM

Here I describe the solution concept used for the analysis of equilibria in large elections. I begin by considering the (more interesting) case where turnout is incomplete, before considering cases where there is complete turnout amongst one faction of voters.

Incomplete Turnout. From equation (3), an equilibrium with incomplete turnout ($t_L \in (0,1)$ and $t_R \in (0,1)$) is characterized by the equalities

$$\Pr[\text{pivotal} \mid R] = \Pr[\text{pivotal} \mid L] = \frac{c}{u}. \tag{16}$$

These probabilities are complicated. Using Lemma 3, however, the approximations

$$\Pr[\text{pivotal} \mid L] \approx \frac{f(p^\star)}{\bar{a}(t_L + t_R)n}\frac{1-p^\star}{1-\bar{p}} \quad \text{and} \quad \Pr[\text{pivotal} \mid R] \approx \frac{f(p^\star)}{\bar{a}(t_L + t_R)n}\frac{p^\star}{\bar{p}} \tag{17}$$

work well when the electorate is large. I proceed, then, in a pragmatic way by assuming that voters employ the approximations in (17) when they evaluate their decisions.

Definition (Solution Concept)**.** *A* voting equilibrium *is a pair of voting probabilities t_L and t_R such that voters act optimally given that they use the asymptotic approximations of* (17).

This is an "ε equilibrium" in the sense that voters are only approximately optimizing. Nevertheless, for moderate electorate sizes the approximations in (17) are good.[7] In Section 8 I consider another justification based upon the solution concept used (for strategic voting) by Myatt (2007) and Dewan and Myatt (2007) and (for protest voting) by Myatt (2015).

I write $\Pr^\dagger[\text{pivotal} \mid i]$ for $i \in \{L, R\}$ for the approximations of (17). If turnout is incomplete then a voting equilibrium must satisfy $\Pr^\dagger[\text{pivotal} \mid L] = \Pr^\dagger[\text{pivotal} \mid R] = (c/u)$. Inspecting (17), notice that the equality of the pivotal probabilities holds if and only if $p^\star = \bar{p}$.

Lemma 4 (Underdog Effect)**.** *In an equilibrium with incomplete turnout: $p^\star \equiv t_L/(t_L + t_R) = \bar{p}$.*

This says that the turnout rates amongst the two factions must exactly offset the prior expected asymmetry between their sizes. Recall that $p^\star \equiv t_L/(t_L + t_R)$ is a critical threshold in the sense that the true popularity of R needs to exceed $p^\star$ if she is to win, at least in expectation. Lemma 4 reveals that a candidate's true popularity must exceed her perceived popularity if she is going to carry the election. (In Section 5 I show that turnout rates only partially offset the prior expected asymmetry when voting costs are heterogeneous.)

Lemma 4 characterizes the relative size of the turnout rates t_L and t_R by solving the equation $\Pr^\dagger[\text{pivotal} \mid L] = \Pr^\dagger[\text{pivotal} \mid R]$. However, it does not tie down the level of these rates. This second step may be performed via the equation $\Pr^\dagger[\text{pivotal} \mid i] = (c/v)$. Before doing this, it is useful to recall that $\bar{t} = \bar{a}[\bar{p}t_R + (1-\bar{p})t_L]$ is the expected turnout rate. Dividing this by

[7] The approximations in (17) are obtained by averaging out the idiosyncratic noise. The law of large numbers bites quickly as n increases, and so aggregate-level uncertainty dominates even for moderate electorate sizes.

$t_L + t_R$, applying Lemma 4, and using the approximations of (17),

$$\frac{\bar{t}}{\bar{a}(t_L+t_R)} = \frac{\bar{p}t_R + (1-\bar{p})t_L}{t_L+t_R} = 2\bar{p}(1-\bar{p}) \quad \Rightarrow$$

$$\Pr{}^{\dagger}[\text{pivotal} \mid L] = \Pr{}^{\dagger}[\text{pivotal} \mid R] = \frac{f(\bar{p})}{\bar{a}n(t_L+t_R)} = \frac{2\bar{p}(1-\bar{p})f(\bar{p})}{n\bar{t}}. \quad (18)$$

Equating this final expression to the cost-benefit ratio (c/u) pins down the equilibrium.

Proposition 1 (Equilibrium). *If (c/u) is not too small then there is a unique equilibrium in which*

$$t_L = \frac{\bar{p}f(\bar{p})u}{\bar{a}nc} \quad \textit{and} \quad t_R = \frac{(1-\bar{p})f(\bar{p})u}{\bar{a}nc}. \quad (19)$$

The asymmetric turnout rates offset any difference in the candidate's perceived popularities: the less popular candidate enjoys greater turnout, and so $\mathrm{E}[v_L] = \mathrm{E}[v_R]$. The expected turnout rate $\bar{t} = 2\bar{p}(1-\bar{p})f(\bar{p})u/cn$ is increasing in the importance of the election u and decreasing in the voting cost c. Fixing $f(\bar{p})$, turnout increases as the expected popularity difference falls.

The final prediction holds because $\bar{p}(1-\bar{p})$ peaks at $\bar{p} = \frac{1}{2}$: turnout is higher in closely fought contests. The effect is weak when the candidates are evenly matched: beginning from $\bar{p} = \frac{1}{2}$, a local change in $\bar{p}$ has only a second-order effect.[8]

The other properties of a voting equilibrium are unsurprising. In particular, the turnout rate is, other things equal, inversely proportional to the electorate's size. However, the "other things equal" is critical: as the electorate size grows, then so may the payoff u which an instrumental voter enjoys from changing the identity of the winner. Also, turnout depends on the density $f(\bar{p})$ of beliefs about p. I consider this in the next section; however, it is worth noting that pre-election information and so $f(\cdot)$ may also be different in larger electorates.

A further observation is that the expected turnout rate $\bar{t}$ does not depend on $\bar{a}$. Inspecting the solutions for t_L and t_R, this is because the turnout rates of those who are "playing the turnout game" rise as $\bar{a}$ falls. This implies that the solution for turnout is robust to the supposition that some voters (a fraction $1-\bar{a}$ in expectation) have decided that their votes cannot count; the behavior of the "real players" endogenously adjusts.[9]

A final observation is that Proposition 1 imposes the condition that the cost of voting is not too small. An equilibrium exhibits incomplete turnout from both factions if and only if $\max\{t_L, t_R\} < 1$. Applying the solutions from (19), this holds if and only if

$$c > \frac{u \max\{\bar{p}, (1-\bar{p})\} f(\bar{p})}{\bar{a}n}. \quad (20)$$

[8]Some have noted (Grofman, 1993) that the claim that "turnout will be higher the closer the election" is "not strongly supported" by the evidence. The claim is weakly supported here, but it should not necessarily be "strongly supported" owing to the second-order size of the effect close to $\bar{p} = \frac{1}{2}$.

[9]This is true only so long as there is an equilibrium with incomplete turnout. Such an equilibrium exists only if (20) holds, and so $\bar{a}$ needs to be large enough. If $\bar{a}$ is sufficiently small (perhaps the "voting is worthless" message has taken hold) then the inequality fails. If this happens, then a voting equilibrium involves incomplete turnout only on one side (the side with the perceived advantage) and complete turnout (amongst those voters who are willing and able to show up) on the other side.

This fails when the election is important (so that u is large); when the cost of voting is small; when the electorate is small; when relatively few are willing to contemplate participation (that is, when $\bar{a}$ is low); and when one candidate is perceived to enjoy a strong advantage.

Complete Turnout for the Underdog Candidate. If (20) fails, then there will be complete turnout on at least one side. The faction with the less popular candidate (in expectation) will be one of those that sees maximal turnout amongst its members.

Lemma 5 (Underdog Effect with Complete Turnout)**.** *Assume (without loss of generality) that R has greater perceived popularity, so that $\bar{p} > \frac{1}{2}$. An equilibrium satisfies $t_R \leq t_L$ so that there is greater turnout for the underdog. If $t_R < 1$ then this holds strictly: $t_R < t_L$. If $t_L = 1$ then $p^\star \leq \bar{p}$.*

Recall that $p^\star$ is the critical threshold which the true popularity of R needs to exceed if she is to win (at least in expectation). If $p^\star < \bar{p}$ then (using Lemma 3) the supporters of R have a weaker incentive to participate, and so $\Pr^\dagger[\text{pivotal} \mid R]$ is the critical factor in any equilibrium. For complete turnout ($t_L = t_R = 1$, so that $p^\star = \frac{1}{2}$), the necessary inequality is $\Pr^\dagger[\text{pivotal} \mid R] \geq (c/v)$ or equivalently $(p^\star)^2 f(p^\star)/(\bar{a}\bar{p}n) \geq (c/u)$ for $p^\star = \frac{1}{2}$. For an equilibrium with complete turnout on one side (so that $t_L = 1$ but $t_R < 1$), the equilibrium is pinned down by the equation $\Pr^\dagger[\text{pivotal} \mid R] = (c/v)$. This equation reduces to

$$\frac{(p^\star)^2 f(p^\star)}{\bar{p}\bar{a}n} = \frac{c}{u} \quad \text{where} \quad p^\star = \frac{1}{1+t_R}. \tag{21}$$

Looking for a solution $t_R \in [0,1]$ is equivalent to seeking a solution $p^\star$ satisfying $\frac{1}{2} \leq p^\star \leq \bar{p}$. If $f(\cdot)$ has a unique mode at $\bar{p}$ then there is at most one solution to equation (21). More generally, multiple solutions are avoided so long as $p^2 f(p)$ is increasing for $p < \bar{p}$; this weaker condition is (as I show in the next section) easily satisfied. Imposing this regularity condition is enough to pin down a unique equilibrium for all cases.[10]

Proposition 2 (Equilibrium with Complete Turnout)**.** *Assume (without loss of generality) that R has greater perceived popularity, so that $\bar{p} > \frac{1}{2}$. If $p^2 f(p)$ is increasing for $p < \bar{p}$ then there is a unique voting equilibrium. If (c/u) is large enough then there is incomplete turnout from both sides. If (c/u) is small enough, then there is complete turnout. For intermediate values of (c/u), however, there is complete turnout for the underdog but only partial turnout by the leader's supporters.*

4. Voters' Beliefs and Predicted Turnout Rates

The properties of beliefs about the candidates' popularity, determined by $f(\cdot)$, are important for turnout. Here I impose more structure on these beliefs, and so relate turnout to voters' knowledge of the electoral situation. I then use a more complete expression for turnout to offer a calibrated prediction of reasonable turnout in a moderately large electorate.

[10] If $p^2 f(p)$ is non-monotonic then there can be multiple equilibria involving complete turnout for candidate L but only partial turnout for candidate R. Nevertheless, even in this case Proposition 1 continues to hold: if (c/u) is not too small then there is a unique equilibrium involving incomplete turnout for both candidates.

Beliefs about the Candidates' Popularities. The density in the solution for $\bar{t}$ is evaluated at the expectation $\bar{p}$. For a well-behaved density this expectation is close to the mode, which helps to maximize the turnout rate. To check this, here I place more structure on $f(\cdot)$.

A natural specification is for p to follow a Beta distribution with parameters β_R and β_L:

$$f(p) = \frac{\Gamma(\beta_R + \beta_L)}{\Gamma(\beta_R)\Gamma(\beta_L)} p^{\beta_R - 1} p^{\beta_L - 1}, \tag{22}$$

where $\Gamma(\cdot)$ is the Gamma function. A special case is when $f(p)$ is uniform: $\beta_R = \beta_L = 1$. The Beta is conveniently conjugate with the binomial distribution. If a voter begins with a uniform prior over p and observes a random sample containing $\beta_R - 1$ supporters of R and $\beta_L - 1$ supporters of L, then his posterior follows the Beta with parameters β_R and β_L. Thus $s = \beta_R + \beta_L$ indexes the size of the sample (allowing for information contained in the prior, together with the actual sample of size $s - 2$) used by a voter to form beliefs.

The mean of the Beta is $\bar{p} = \beta_R/(\beta_R + \beta_L)$. The density may be written in terms of $\bar{p}$ and a parameter s which corresponds to the information available to a voter; as explained above, $s - 2$ would corresponds to the sample size of an opinion poll, yielding an effective precision proportional to s once the prior is taken into account. Using this formulation,

$$f(p) = \frac{\Gamma(s)}{\Gamma(\bar{p}s)\Gamma((1-\bar{p})s)} p^{\bar{p}s - 1}(1-p)^{(1-\bar{p})s-1}. \tag{23}$$

$p^2 f(p)$ is increasing for $p < \bar{p}$, and this meets the condition of Proposition 2; there is a unique equilibrium. The density $f(p)$ can be substituted into the turnout solution. Doing so:

$$\bar{t} = \frac{\Gamma(s)}{\Gamma(\bar{p}s)\Gamma((1-\bar{p})s)} \frac{2u[\bar{p}^{\bar{p}}(1-\bar{p})^{(1-\bar{p})}]^s}{cn}. \tag{24}$$

To see things a little more clearly, and when s is large enough, the Beta density can be approximated with a normal distribution. The variance of the Beta, in terms of s and the mean $\bar{p}$, satisfies $\mathrm{var}[p] = \bar{p}(1-\bar{p})/(s+1)$. So, using a normal approximation,

$$f(p) \approx \sqrt{\frac{s+1}{2\pi\bar{p}(1-\bar{p})}} \exp\left(-\frac{(s+1)(p-\bar{p})^2}{2\bar{p}(1-\bar{p})}\right), \tag{25}$$

where here π indicates the mathematical constant and not a model parameter. When evaluated at $\bar{p}$ the exponential term disappears, so generating the next result.

Proposition 3 (Equilibrium with Beta Beliefs)**.** *Using a Beta specification for voters' beliefs (interpreted as the common public posterior belief following the publication of an opinion poll) there is a unique voting equilibrium. If (c/u) is not too small, this equilibrium involves incomplete turnout. Using a normal approximation for voters' beliefs, expected turnout satisfies*

$$\bar{t} = \frac{u\bar{p}(1-\bar{p})}{cn}\sqrt{\frac{2}{\pi\,\mathrm{var}[p]}}. \tag{26}$$

This is increasing in the precision of voters' beliefs about the candidates' popularity.

Equation (26) predicts that turnout is greatest when the election is important; when costs are low; when candidates are evenly matched; when (all else equal) the electorate is smaller; and, finally, when there is good information about the popularities of the candidates.[11]

Calibration. It is often suggested that rational-choice theory predicts low turnout. If turnout were high then the probability of a tie in a large electorate is (it is claimed by some) far too low to justify the cost of voting. For example Green and Shapiro (1994, Chapter 4) claimed:

> Although rational citizens may care a great deal about ... the election, an analysis of the instrumental value of voting suggests that they will nevertheless balk at ... contributing to a collective cause since it is readily apparent that any one vote has an infinitesimal probability of altering the election outcome.

Other things equal, the probability of a tie does decline as the electorate size grows. This, however, does not justify the "too low to vote" conclusion. It is not clear that the probability is "infinitesimal" and it is not "readily apparent" that there is no hope for an instrumental explanation for the turnout decision. What is needed is an assessment of precisely how big or small the pivotal probability is. To move forward I proceed with a calibration exercise: I choose reasonable parameters and ask whether plausible levels of turnout emerge.

I begin with the precision of beliefs. In the context of Proposition 3, the variance $\mathrm{var}[p]$ can be used to construct the width Δ of a confidence interval regarding the popularity of candidate R. Familiar calculations from classical statistics yield $\Delta \approx 3.92 \times \sqrt{\mathrm{var}[p]}$ for an interval at the usual 95% level. Using equation (26) from Proposition 3 with $\bar{a} = 1$,

$$\bar{t} = \frac{u\bar{p}(1-\bar{p})}{cn}\sqrt{\frac{2}{\pi\,\mathrm{var}[p]}} \approx \frac{u\bar{p}(1-\bar{p})}{cn}\frac{3.92}{\Delta}\sqrt{\frac{2}{\pi}} \approx 3.13 \times \frac{u\bar{p}(1-\bar{p})}{cn\Delta}. \tag{27}$$

Next, I write the turnout rate in terms of the population size N rather than the electorate size n. For example, just under 75% of the United Kingdom's population are registered to vote, and so I set $n = 0.75 \times N$. Arguably this is generous, and so works against a high turnout rate; for instance, in the United States the electorate is a smaller fraction. Nevertheless,

$$\bar{t} \approx 3.13 \times \frac{u\bar{p}(1-\bar{p})}{0.75 \times cN\Delta} \approx 4.17 \times \frac{u\bar{p}(1-\bar{p})}{cN\Delta}. \tag{28}$$

Finally, I pick a value for the expected popularity of the leading candidate. Obviously, if the candidates are seen as evenly matched then turnout is higher. So, to work against higher turnout I choose a more unbalanced $60 : 40$ split, so that $\bar{p} = 0.6$. Doing so,

$$\bar{t} \approx 4.17 \times \frac{u \times 0.6 \times 0.4}{cN\Delta} = 4.17 \times 0.24 \times \frac{u}{cN\Delta} \approx \frac{(u/c)}{N\Delta}. \tag{29}$$

I record this calibration exercise as a simple proposition. This propositions provides the support for the numerical vignette used within the introductory remarks to the paper.

[11] The final prediction of Proposition 3 is supported by established empirical work. Gentzkow (2006) used between-market variation in the timing of the introduction of television to identify an negative effect on turnout. The introduction of television "caused sharp drops in consumption of newspapers and radio" and "reduced citizens' knowledge of politics as measured in election surveys" (Gentzkow, 2006, p. 932). This switch away from other media, which in turn reduced the extent of electoral coverage, particularly in off-year congressional elections, is consistent with an increase in $\mathrm{var}[p]$ and so a fall in turnout rates.

Proposition 4. *Consider a region in which 75% of the population are eligible to vote, where a 95% confidence interval for popularity of the leading candidate is centered at 60%. Then,*

$$\textit{expected turnout rate} \approx \frac{\textit{instrumental benefit / voting cost}}{\textit{population} \times \textit{width of 95\% confidence interval}}. \tag{30}$$

Hence, for 100,000 people (such as Cambridge; either Massachusetts or England), if a confidence interval for the more popular candidate ranges from 57% to 62% (following a typical opinion poll), and if voters are willing to participate for a 1-in-2,500 influence, then turnout should exceed 50%.

In Section 8 I consider further calibration exercises for much larger electorates (at the scale of a larger city, state, or country) when voters have other-regarding preferences.

5. Asymmetric and Idiosyncratic Voting Costs

Here I extend the model by varying the cost c of voting relative to the payoff parameter u.

Asymmetric Voting Costs. I begin by allowing the participation cost to differ between the two factions. Using obvious notation, an equilibrium with incomplete turnout satisfies

$$\Pr{}^{\dagger}[\text{pivotal} \mid L] = \frac{c_L}{u} \quad \text{and} \quad \Pr{}^{\dagger}[\text{pivotal} \mid R] = \frac{c_R}{u}. \tag{31}$$

Lemma 3 (concerning pivotal probabilities in larger electorates) holds. However, Lemma 4 does not: the critical threshold $p^\star = t_L/(t_L+t_R)$ for R's popularity does not necessarily equal the prior expectation $\bar{p}$. Instead, combining the conditions from equation (31) yields

$$p^\star = \frac{\bar{p}c_R}{\bar{p}c_R + (1-\bar{p})c_L}. \tag{32}$$

Suppose (without loss of generality) that R is more popular ex ante, so that $\bar{p} > \frac{1}{2}$. By inspection, $p^\star > \bar{p} > \frac{1}{2}$ if and only if $c_R > c_L$. That is, if the supporters of R find it (relatively) more costly to vote then disproportionately higher turnout for L more than offsets the popularity advantage which R enjoys. Overall, a randomly chosen voter who shows up at the polling booth is more likely to vote for L (since $\bar{p}t_R < (1-\bar{p})t_L$).

Proposition 5 (Asymmetric Voting Costs)**.** *Suppose that types L and R face different costs of voting. In an equilibrium with incomplete turnout (this is unique if u is not too large):*

$$t_L = \frac{uf(p^\star)p^\star(1-p^\star)}{c_L\bar{a}n(1-\bar{p})} \quad \textit{and} \quad t_R = \frac{uf(p^\star)p^\star(1-p^\star)}{c_R\bar{a}n\bar{p}}. \tag{33}$$

The candidate with lower-cost voters enjoys greater expected support: $\bar{p}t_R > (1-\bar{p})t_L \Leftrightarrow c_R < c_L$. Using a normal specification for $f(\cdot)$, turnout is non-monotonic (first increasing, then decreasing) in the precision of beliefs. The expected turnout rate falls to zero as beliefs become very precise.

The effect of the precision of beliefs (measured by $1/\operatorname{var}[p]$) on turnout is a feature here. If voters share the same costs then expected turnout increases as beliefs become more precise (Proposition 1). The asymmetry in voting costs (relative to u; asymmetric costs are equivalent to specifying factions with different strengths of feeling) overturns this. This indicates that there may be some fragility in models which generate high turnout using an "IID" specification in which the popularity of candidates is known and payoffs are symmetric.

Idiosyncratic Voting Costs. Next I consider an environment in which voting costs are idiosyncratic (there is heterogeneity within the factions) but where there is no systematic difference between the fractions. Voters' costs are independently drawn from a known distribution, and a voter's cost is independent of his preference type. For $t \in [0,1]$, I write $C(t)$ for the inverse of the distribution function of voting costs, so that $t = \Pr[c \leq C(t)]$, and I make three regularity assumptions: $C(t)$ is strictly and continuously increasing; $C(0) = 0$; and $C(1)$ is large enough to ensure incomplete turnout from both sides.

If voter types turn out with probabilities t_L and t_R then the costs of the marginal participating voters are $C(t_L)$ and $C(t_R)$ respectively. The two equalities satisfied are simply

$$\Pr^{\dagger}[\text{pivotal} \mid L] = \frac{C(t_L)}{u} \quad \text{and} \quad \Pr^{\dagger}[\text{pivotal} \mid R] = \frac{C(t_R)}{u}. \tag{34}$$

Taking ratios and using Lemma 3 yields $\bar{p} t_R C(t_R) = (1-\bar{p}) t_L C(t_L)$. A recurring theme that the turnout rate is higher for the underdog: if $\bar{p} > \frac{1}{2}$ then $t_L > t_R$. However, the presence of the $C(\cdot)$ terms ensure that higher turnout is not enough to offset completely a popularity disadvantage; if $\bar{p} > \frac{1}{2}$ then the critical popularity threshold $p^\star$ satisfies $\frac{1}{2} < p^\star < \bar{p}$.[12] If R is less popular than she is expected to be (so that $p < \bar{p}$) then she may still win (if $p > p^\star$) but also she may lose despite being the more popular candidate; this happens when $\frac{1}{2} < p < p^\star$.

The equilibrium is easily characterized when costs are uniformly distributed. If $c \sim U[0,1]$ then $tC(t) = t^2$, and the equality $\bar{p} t_R C(t_R) = (1-\bar{p}) t_L C(t_L)$ yields

$$\frac{p^\star}{1-p^\star} = \sqrt{\frac{\bar{p}}{1-\bar{p}}}. \tag{35}$$

Hence, under the uniform specification the relative turnout of the two factions is uniquely determined by the expected popularity of one candidate relative to the other; the other parameters of the model have no major role to play. These observations, together with the effect of the precision of voters' beliefs, are summarized in the following proposition.

Proposition 6 (Equilibrium with Idiosyncratic Voting Costs)**.** *If voting costs vary then there is higher turnout from supporters of the less popular candidate, but this does not offset her expected disadvantage: if $\bar{p} > \frac{1}{2}$ then $t_L > t_R$ but $\frac{1}{2} < p^\star < \bar{p}$. If voting costs are uniformly distributed then*

$$t_L = \sqrt{\frac{uf(p^\star)p^\star(1-p^\star)}{\bar{a}n(1-\bar{p})}} \quad \textit{and} \quad t_R = \sqrt{\frac{uf(p^\star)p^\star(1-p^\star)}{\bar{a}n\bar{p}}} \quad \textit{where} \quad p^\star = \frac{\sqrt{\bar{p}}}{\sqrt{\bar{p}} + \sqrt{1-\bar{p}}}. \tag{36}$$

Relative turnout is independent of the availability of voters, of the precise nature of voters' prior beliefs $f(\cdot)$, of the electorate size, and of the importance of the election. Additionally, with a normal approximation for $f(\cdot)$, turnout is non-monotonic in the precision of beliefs: it is first increasing and then decreasing in $1/\operatorname{var}[p]$, falling to zero as beliefs become arbitrarily precise.

Turnout eventually falls as the precision of beliefs increases (just as it did as a conclusion of Proposition 5) because $p^\star \neq \bar{p}$ and so the density $f(\cdot)$ is evaluated away from $\bar{p}$. As beliefs become very precise, the density clumps around $\bar{p}$, and so the density elsewhere falls.

[12] This is the "partial underdog compensation effect" highlighted by Herrera, Morelli, and Palfrey (2014).

It is also interesting to look at the expected turnout rate when $c \sim U[0,1]$. This is:

$$\bar{t} = \sqrt{\frac{\bar{a}u\sqrt{\bar{p}(1-\bar{p})}}{n} f\left(\frac{\sqrt{\bar{p}}}{\sqrt{\bar{p}}+\sqrt{1-\bar{p}}}\right)}. \tag{37}$$

The turnout rate now inversely related to the square root of the electorate size; this implies that the expected total turnout is increasing with n. This contrasts with the homogenous-costs case, where it is independent of n. Note also that the turnout rate is increasing in the expected availability of voters. These features are because voting costs are heterogeneous with a support that extends down to zero. (Recall that I am using a uniform distribution here, so that $c \sim U[0,1]$.) Those who turn out are those with the lowest costs, and there are simply more of them as the electorate size and voters' availability both grow.

The Underdog Effect. A message from this section is that endogenous turnout causes the familiar effect of enhancing the chances of the underdog; indeed, it generates situations in which the a more popular candidate loses. However, only when payoffs are common to all electors is this effect so strong as to offset exactly any prior asymmetry in popularity. With intra-faction variation in costs, there is only a partial pro-underdog bias in turnout.

6. DOES VOLUNTARY PARTICIPATION SELECT THE RIGHT WINNER?

The underdog effect biases against the perceived leader. Hence, if this candidate is the true leader but less popular than she is perceived to be (so that $\bar{p} > p > \frac{1}{2}$ when the leader is R) then the election typically picks the wrong winner. This evaluation requires a definition of what it means (from a social viewpoint) to pick the correct candidate. I do that here, before asking whether a benevolent planner would wish to hold the election. I also investigate the impact of asymmetric payoffs and idiosyncratic turnout costs.

Electoral Performance. For simplicity I evaluate the aggregate outcome conditional on p, the underlying probability that a voter prefers candidate R. (The realized proportion who prefer R converges in probability to p as n grows.) The net gain from a win for R relative to a win for L is $pu-(1-p)u=(2p-1)u$, and so R as the right winner if and only if $p > \frac{1}{2}$. I focus on "picking the right winner" and so I put aside voting costs. I also suppose that the electorate size and turnout rates are both large, and so I assume that R wins (as she does with high probability) if and and only if $p > p^\star$. Setting $u = 1$ without loss of generality, my social performance measure W is the proportion of voters who get what they want:

$$W = \begin{cases} p & \text{if } p > p^\star\text{, or} \\ (1-p) & \text{if } p < p^\star. \end{cases} \tag{38}$$

Given voluntary turnout, the expected performance $\mathrm{E}[W] = \int_0^{p^\star}(1-p)f(p)\,dp + \int_{p^\star}^1 pf(p)\,dp$ is maximized when $p^\star = \frac{1}{2}$. I have already demonstrated that $p^\star = \bar{p}$. Allowing R to be the more popular, so that $\bar{p} > \frac{1}{2}$, this means that there is a bias (owing to the underdog effect) toward the less popular competitor. This weakens the social performance of a voluntary-turnout electoral system. But what is the alternative?

To answer, consider a benevolent planner who either chooses the winner without reference to voters or, alternatively, opts for an election with voluntary turnout. The election allows the choice to respond to the true underlying popularities, but it biases against the expected leader. If $\bar{p} = \frac{1}{2}$ then (putting aside voting costs) the answer is straightforward: the planner should hold the election. If $\bar{p}$ deviates from $\frac{1}{2}$ then two things happen. Firstly, the bias toward the underdog strengthens, which weakens the performance of the election; and, secondly, the stronger candidate becomes clearer ex ante and so the need for an election is weakened.

To illustrate this these effects, suppose that $\bar{p} > \frac{1}{2}$ so that a planner would optimally choose candidate R and social performance (the expected proportion of citizens who get what they want) would be $\bar{W} = \bar{p}$. The net gain from holding the election is

$$\mathrm{E}[W] - \bar{W} = \int_0^{\bar{p}} (1 - 2p) f(p)\, dp. \tag{39}$$

A change in the distribution of p affects this in two ways: via the density $f(p)$ and via a change in $\bar{p}$. This second effect is negative when $\bar{p} > \frac{1}{2}$ owing to the enhanced underdog effect. Obviously more structure is needed to ascertain the effect of a change in $f(p)$. Consider, for example, the case where $p \sim U[\bar{p} - \sigma, \bar{p} + \sigma]$. For this case

$$\mathrm{E}[W] - \bar{W} = \frac{1 + \sigma}{2} - \bar{p}. \tag{40}$$

By inspection, the desirability of the voluntary-participation election falls as $\bar{p}$ increases and σ falls. If there is confidence that one candidate has a strong expected lead then (given the underdog effect on relative turnout) the election too often picks the wrong candidate.

Proposition 7 (Holding the Election)**.** *Consider a social planner who wishes to maximize the probability that the chosen candidate is the favorite candidate of a randomly chosen voter. Adopt the assumptions of this section, and assume that $p \sim U[\bar{p} - \sigma, \bar{p} + \sigma]$. If*

$$\frac{1}{2} - \sigma < \bar{p} < \frac{1}{2} + \sigma \tag{41}$$

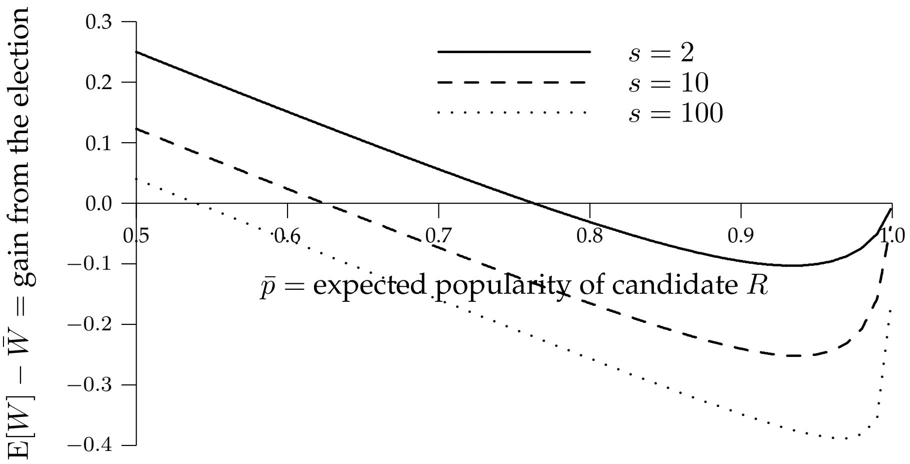
then the planner prefers to hold the election with voluntary participation, but otherwise prefers to choose directly the candidate with the greatest expected popularity. The election is socially desirable if (i) the candidates are relatively evenly matched and if (ii) their popularities are more uncertain.

The same logic applies to the Beta distribution that I used in Section 4. Recall from there that

$$f(p) = \frac{\Gamma(s)}{\Gamma(\bar{p}s)\Gamma((1 - \bar{p})s)} p^{\bar{p}s - 1} (1 - p)^{(1 - \bar{p})s - 1}, \tag{42}$$

Figure 1 illustrates the gain from holding an election as a function of $\bar{p}$ and s. Once again, an election is desirable when the identity of the true leader is not too clear.

Asymmetric Strengths of Feeling. So far I have assumed that members of the two factions share the same strength of feeling about their favorites. I now allow for inter-factional variations in this strength of feeling via (using obvious notation) payoff parameters u_L and u_R. This payoff asymmetry has a similar effect to the cost asymmetry considered previously (in



Notes. This figure uses the Beta specification for beliefs from eq. (42). It illustrates the net gain (measured via the expected match between the chosen candidate and a random voter's preference) from operating a voluntary-turnout election versus choosing (without reference to any ballot) the candidate with the greatest expected popularity.

FIGURE 1. The Performance of a Voluntary-Turnout Election

Section 5). For an equilibrium with incomplete turnout, the turnout rates satisfy

$$t_L = \frac{u_L f(p^\star) p^\star (1 - p^\star)}{c\bar{a}n(1 - \bar{p})} \quad \text{and} \quad t_R = \frac{u_R f(p^\star) p^\star (1 - p^\star)}{c\bar{a}n\bar{p}}. \tag{43}$$

The ratio of these turnout rates determines the critical popularity that R needs win:

$$\frac{p^\star}{1 - p^\star} = \frac{t_L}{t_R} = \frac{u_L}{u_R} \times \frac{\bar{p}}{1 - \bar{p}}. \tag{44}$$

Just as before (when strengths of feeling were equal) the underdog effect is present: candidate R is disadvantaged by higher expected popularity. However, this underdog effect does not apply to the strength-of-feeling parameters: relative turnout scales up with this feeling. Hence, if a faction's members feel twice as strongly about the election then (other things equal) turnout from that faction will be twice as high.

Turning to the planner, consider again a social performance criterion which seeks to maximize (in a large electorate) expected payoff of a randomly chosen citizen. In this case

$$W = \begin{cases} p u_R & \text{if } p > p^\star \text{, or} \\ (1 - p) u_L & \text{if } p < p^\star. \end{cases} \tag{45}$$

The planner prefers to see a win by R if and only if $p > p^\ddagger$ where $p^\ddagger/(1-p^\ddagger) = u_L/u_R$. If $\bar{p} > \frac{1}{2}$ (so that R is the expected leader) then $p^\ddagger < p^\star$. Equivalently, the bias against the apparent leader is maintained here. In fact, with asymmetric payoff parameters it is straightforward to generalize Proposition 7: once again, an election is (socially) desirable when the candidates are expected to be close and when prior beliefs are not too precise.

Idiosyncratic Voting Costs. In Section 5 I showed that the pro-underdog bias is weakened when voting costs are idiosyncratic. This can lessen the risk of electing the wrong winner.

To illustrate this, I assume again that a turnout rate of t is associated with a marginal turnout cost of $C(t)$. Allowing for asymmetric strengths of feeling, the equalities from (34) become

$$\Pr^{\dagger}[\text{pivotal} \,|\, L] = \frac{C(t_L)}{u_L} \quad \text{and} \quad \Pr^{\dagger}[\text{pivotal} \,|\, R] = \frac{C(t_R)}{u_R}. \tag{46}$$

Just as before, these can be combined and used with Lemma 3. Doing so, the equation

$$\frac{t_L C(t_L)}{t_R C(t_R)} = \frac{u_L}{u_R} \times \frac{\bar{p}}{1-\bar{p}} \tag{47}$$

pins down the ratio of the turnout rates on both sides.

Consider a marginal turnout cost function that satisfies $C(t) = t^{\delta}$ for some $\delta > 0$. For example, if $\delta = 1$ then $C(t) = t$ which is equivalent to $c \sim U[0,1]$. For other values of δ, and writing $k(c)$ for the density function of the turnout cost $c \in [0,1]$, it is equivalent to setting

$$k(c) = \frac{c^{(1-\delta)/\delta}}{\delta}. \tag{48}$$

This density is increasing in δ (so approximating the left-tail of a unimodal distribution) if and only if $\delta < 1$. As already noted, the uniform is obtained for $\delta \to 1$. Allowing $\delta \to 0$ shifts all mass to $c = 1$, and so this limiting case corresponds to homogeneous costs.

With this formulation, (47) solves to yield

$$\frac{p^{\star}}{1-p^{\star}} = \frac{t_L}{t_R} = \left[\frac{u_L}{u_R} \times \frac{\bar{p}}{1-\bar{p}}\right]^{1/(1+\delta)}. \tag{49}$$

Just as before, there is a bias against the perceived leader: by inspection, $p^{\star}$ is increasing in $\bar{p}$ and so greater expected popularity for candidate R shifts turnout against her.

Putting popularities aside, the turnout rates of the factions also react to preference intensities. This is welcome: a benevolent planner would like turnout to reflect strengths of feeling. However, this reaction is more sluggish than it is in a homogenous-cost environment. For example, if $\delta = 1$ (the case where $c \sim [0,1]$) then quadrupling the strength of feeling for a candidate only doubles her turnout relative to the turnout of her competitor.

To see why there is an under-reaction to strength of feeling, consider the case where $\bar{p} = \frac{1}{2}$ (so that there is no underdog, and so no pro-underdog bias) but where $u_R > u_L$. This induces greater turnout for R. However, this means that the marginal participant from the right-wing faction has a higher turnout cost. This works against turnout from this faction, and so partially offsets the effect of greater strength of feeling.

The underdog effect (lower turnout for the perceived populist) and the under-reaction effect (the insufficient response to greater strength of feeling) both distort the key threshold $p^{\star}$ away from the right winner. (In fact the underdog effect is also an under reaction: the actual votes for a candidate under-react to that candidate's popularity.) Just as before, candidate R is the right winner if $p > p^{\ddagger}$ where $p^{\ddagger}/(1-p^{\ddagger}) = u_L/u_R$ as before. The election is biased against candidate R if and only if $p^{\star} > p^{\ddagger}$ which holds if and only

$$\frac{u_L}{u_R} < \left[\frac{u_L}{u_R} \times \frac{\bar{p}}{1-\bar{p}}\right]^{1/(1+\delta)} \quad \Leftrightarrow \quad \left[\frac{u_L}{u_R}\right]^{\delta} < \frac{\bar{p}}{1-\bar{p}}. \tag{50}$$

There is a bias unless $\bar{p}/(1-\bar{p}) = [u_L/u_R]^\delta$. This typically fails, and so in general a voluntary-turnout election discriminates against the leading candidate: note that the inequality is most easily satisfied when $\bar{p}$ is large (a candidate is expected to be popular) and when u_R is relatively large (the candidate's supporters have strong preferences).

If turnout rates are biased against a candidate then the election sometimes picks the wrong winner. However, this matters only if the situations in which the election gets its wrong are likely. To illustrate this I now consider a case in which the electoral situation is very clear. Specifically, suppose that the true popularity of candidate R satisfies $p = \bar{p}$. Without loss of generality set $\bar{p} > \frac{1}{2}$ so that candidate R is the perceived popular leader. There is a meaningful bias against her, in the sense that she loses if though she should have won, if

$$p^\dagger < \bar{p} < p^\star \quad \Leftrightarrow \quad \frac{u_L}{u_R} < \frac{\bar{p}}{1-\bar{p}} < \left[\frac{u_L}{u_R} \times \frac{\bar{p}}{1-\bar{p}}\right]^{1/(1+\delta)}. \tag{51}$$

The performance of an election is a concern only if there are parameter choices for which these inequalities are satisfied. If they can be satisfied then they are also satisfied when the strengths-of-feeling ratio is raised until $(u_L/u_R) = \bar{p}/(1-\bar{p})$ (ensuring that the first inequality holds as an equality) which means that a planner is indifferent ex ante between the candidates. Given that this is so, the second inequality reduces to

$$\left[\frac{\bar{p}}{1-\bar{p}}\right]^\delta < \frac{\bar{p}}{1-\bar{p}}, \tag{52}$$

which (given that $\bar{p} > \frac{1}{2}$) holds if and only if $\delta < 1$. These observations (concerning the election's bias and the circumstances in which it matters) are summarized here.

Proposition 8 (Turnout Bias)**.** *Suppose that voters' participation costs are idiosyncratic, and described by a density $k(c) \propto c^{(1-\delta)/\delta}$ for some $\delta > 0$. The turnout bias against a candidate is increasing in that candidate's perceived popularity and in the strength of her faction's preferences for her.*

Now suppose that $p = \bar{p}$ so that the electoral situation matches expectations. There is a meaningful bias against the popular leader (she sometimes loses even though she should win) if and only if $\delta < 1$.

Of course, if $\delta > 1$ then there is a bias toward the popular leader. The crossover case is when $\delta = 1$, so that the distribution of turnout costs is uniform. In this case (as noted by Proposition 8) there is a bias against the popular leader if and only if she is also the choice of the planner. However, this bias does not really matter in the sense that if a planner is confident that $p = \bar{p}$ then the election's decision coincides with the planner's.

Corollary (to Proposition 8)**.** *Consider the cost specification underpinning Propositions 8. If prior expectations are accurate, so that $p = \bar{p}$, then an election with voluntary turnout always picks the right winner for all u_R and u_L if and only if the participation cost is uniformly distributed.*

The anti-populism bias (against the candidate with the higher perceived popularity) when $\delta < 1$ is more general. Suppose, for examine, that $k(c)$ is a unimodal on $[0, \infty)$ and that turnout rates are such that each marginal voter is below the mode. This implies (over the relevant range) that $C(t)$ is concave. If $t_L > t_R$ then $C(t_L)/C(t_R) < t_L/t_R$, and so

$$\frac{p^\star}{1-p^\star} = \frac{t_L}{t_R} > \sqrt{\frac{t_L C(t_L)}{t_R C(t_R)}} = \sqrt{\frac{u_L \bar{p}}{u_R(1-\bar{p})}}. \tag{53}$$

If a benevolent planner is indifferent ex ante then $u_R\bar{p} = u_L(1-\bar{p})$ which implies that the right-hand side is equal to $\bar{p}/(1-\bar{p})$. Hence $p^\star > \bar{p}$. So, if the true popularity of R matches the expectation then she will be defeated. This illustrates how the wrong winner can be picked.

Proposition 9 (Bias Against the Populist)**.** *Suppose that voters' participation costs are described by a density $k(c)$ that is increasing in c for all $c < \bar{c}$. If marginal voters' turnout costs fall below $\bar{c}$, and if prior expectations are accurate, so that $p = \bar{p}$, then there is a bias against the more popular candidate: the wrong candidate can be elected, and if so then the loser is the popular candidate.*

Election Performance. A message here is that an election with voluntary turnout often picks the wrong winner. If the distribution of voting costs is unimodal, and if the marginal voter from each faction is in the left tail of that distribution (which seems reasonable when turnout falls below 50%) then the bias is against the perceived popular candidate.

This message contrasts with recent papers (Krishna and Morgan, 2011, 2012, 2014, 2015) which claim that voluntary turnout picks the right winner. Krishna and Morgan (2011, 2014) used IID models without aggregate uncertainty and concluded that "when voting is voluntary … turnout adjusts endogenously so that the outcome of a large election is always first-best" (Krishna and Morgan, 2011, p. 183) and similarly "a large election under majority rule produces the utilitarian choice" (Krishna and Morgan, 2014).[13] Their very recent work-in-progress (Krishna and Morgan, 2015) that I have seen presented builds upon ideas from an older version of this paper (Myatt, 2012a) and includes aggregate uncertainty. Here, the equilibrium threshold for a candidate's success (that is, $p^\star$) is almost always wrong (so that $p^\star \neq p^\ddagger$). Nevertheless, if $p = \bar{p}$ then one may still ask: does the voters' choice coincide with the expected utilitarian candidate? Krishna and Morgan (2015) answered positively.

This is not the case here: I find (when $p = \bar{p}$) that the election picks the (expected) utilitarian winner only when voting costs are exactly uniform. In fact, this is the assumption that Krishna and Morgan (2011, 2014, 2015) used. More fully, for larger electorates they specified a distribution of voting costs with a lower bound of support at zero and where (crucially) the density is strictly positive at that lower bound. (The distribution $c \sim U[0,\bar{c}]$ meets this requirement, but $c \sim U[\pm\varepsilon,\bar{c}]$ does not. Moreover, the specification $k(c)$ does not meet this requirement for $\delta < 1$.) For any such distribution the lower tail can be well approximated by a uniform distribution. The Krishna-Morgan papers consider elections with very large electorates ($n \to \infty$) while the stakes (u_R and u_L) are fixed. This generates turnout rates that converge to zero as the electorate grows (although total turnout does not, since the number of voters with negligible voting costs increase). Hence, only the bottom of the tail of the cost distribution matters; and as I have noted here, this is uniform in the very particular case where there is strictly positive density at exactly zero but no density below that. More generally, their claim works for finite n when $c \sim U[0,\bar{c}]$.

[13]Krishna and Morgan (2012) offered a similar message but focused on a common-value jury environment.

The uniformity of costs is crucial. If the density instead declines to zero at zero then the lower tail of the distribution has an increasing density. Proposition 9 reveals that this can (and does) push against the popular leader. Of course, the Krishna-Morgan "large n" results do not require exact uniformity. However, for those results they have relied very strongly on the knife-edge assumptions of a strictly positive density and an exact lower bound of zero.

There is no conflict between my results and those of Krishna and Morgan (2011, 2014, 2015). In essence, they have dealt with the special case where voting costs are uniformly distributed. However, if one accepts that the leading case of interest is one in which a marginal voter sits within the lower tail of unimodal distribution (or indeed not too far up into the upper tail) then perhaps a more reasonable message is this: voluntary voting is not always perfect, and in fact (when $p = \bar{p}$) usually biases against a leading popular candidate.

7. Candidate Positions with Endogenous Turnout

So far I have put aside the strategic actions of candidates: their positions have been implicitly fixed. If everyone participates (for example, under compulsory voting) then the outcome is determined by the position (on a left-right policy spectrum) of the indifferent voter (for instance, the citizen halfway between the candidates' positions) relative to the median voter. The classic prediction is that success is achieved by positioning at the median.

The usual policy-position logic asks how the indifferent voter moves with policy shifts. But if turnout is costly then he and others close to him stay at home. If so, then the determinant of electoral success is the turnout rate for a candidate relative to her opponent. My attention turns, therefore, to the relationship between policy positions and turnout rates.

Popularity versus Preference Intensity. The underdog effect works against the popular candidate. If she could secretly increase her true popularity while keeping expectations fixed then of course she would gain. However, inevitably $\bar{p}$ rises with anything that shifts voters to her, and so any gains (from an increase in p) are offset by dampened turnout.

In contrast, a candidate gains from preference intensity. An increase in u_R (in the asymmetric-strengths-of-feeling model) lowers $p^\star$ and so lowers the bar to success for candidate R. It follows that an office-motivated candidate can do better with a smaller faction of more fanatical supporters rather than relying upon a broader (but less enthusiastic) base.

It may be difficult to change the distribution of voter types. However, their preferences are influenced by policies. A move to the right by candidate R may appeal to her core supporters and so increase u_R. However, it is disliked by opposition voters: the payoff u_L to left-wing voters incorporates a stronger desire to defeat R, and so both sides feel more strongly if a candidate takes a more extreme position. I take up this theme next.

Positioning the Party to Get Out the Vote. I now consider candidates who choose their policy positions. As in Section 1, voters share the same cost of participation and are either right-wing or left-wing types. The two factions are located at either end of a policy spectrum $x \in [0, 1]$. The two candidates locate at positions x_L and x_R where $0 < x_L < x_R < 1$.

A voter's payoff depends upon the identity of the winner and her policy. A right-wing voter receives a payoff $u_R^+(x_R)$ from a win by R or a payoff $u_R^-(x_L)$ for a win by L. Hence, the net gain from a win by R rather than L is $u_R = u_R^+(x_R) - u_R^-(x_L)$. Similarly, the net gain to a left-wing voter from a win by L is $u_L = u_L^+(x_L) - u_L^-(x_R)$. I assume that $u_L^+(x)$ and $u_L^-(x)$ are decreasing in x and similarly $u_R^+(x)$ and $u_R^-(x)$ are increasing in x. If a voter cares only about the implemented policy then $u_R^+(x) = u_R^-(x)$ and $u_L^+(x) = u_L^-(x)$.

If overall turnout is sufficiently strong (for example, when n is large but c is sufficiently small) then candidate R wins (with high probability) if and only if $p > p^\star = t_L/t_R$, and so her success depends on her relative turnout rate. Recalling equation (44) from Section 6:

$$\frac{p^\star}{1-p^\star} = \frac{\bar{p}}{1-\bar{p}} \times \frac{u_L}{u_R} \quad \text{where} \quad \frac{u_L}{u_R} = \frac{u_L^+(x_L) - u_L^-(x_R)}{u_R^+(x_R) - u_R^-(x_L)}. \tag{54}$$

A candidate influences her prospects via policy positions that change u_L/u_R, which she seeks either to minimize (candidate R) or maximize (candidate L).

Beginning from $x_L < x_R$, I now consider the incentive for a candidate to move her position. This depends upon the shape of policy preferences, the presence or absence of policy-independent loyalties, and any valence advantage enjoyed by a candidate.

To pin down the first factor (the shape of preferences) suppose that a voter's payoff depends only on the distance of the winning policy from his ideal point. For some increasing policy loss function $D(\cdot)$, setting $u_L^\pm(x) = \bar{u} - D(x)$ and $u_R^\pm(x) = \bar{u} - D(1-x)$ yields

$$\frac{u_L}{u_R} = \frac{D(x_R) - D(x_L)}{D(1-x_L) - D(1-x_R)}. \tag{55}$$

With symmetric policy positions ($x_R > \frac{1}{2} > x_L$ and $x_L = 1 - x_R$) this ratio is equal to one. An increase in x_R (a rightward shift by R) increases both the numerator and denominator. It increases the numerator by more (and so is undesirable from R's perspective) if and only if $D'(x_R) > D'(1-x_R)$. Given that $x_R > \frac{1}{2} > 1 - x_R$ this holds if the policy-loss function is convex. Similarly, if $D(\cdot)$ is concave then u_L/u_R is locally decreasing in x_R.

Proposition 10 (The Incentive to Change Policy Position)**.** *Adopting the model specification of this section, suppose that each voter's payoff is determined by the distance of the implemented policy from his ideal point. Beginning from symmetric policy positions, the candidates face an incentive to move outward if the policy loss function is concave, and an incentive to move inward if it is convex.*

I have not mapped out a full policy-choice game. Nevertheless such a game is easily constructed. Under appropriate conditions, a concave policy-loss function leads to extremal policies. This concavity corresponds to voters who care most about the initial moves away from their faction's preferred policy. If a candidate steps away from an extreme then she turns off her own supporters by more than she reduces the distaste of her opponents.[14]

A benchmark case (separating those discussed in the proposition) is when the policy-loss function is linear. If this is so then $D(x_R) - D(x_L) = D(1-x_L) - D(1-x_R) \propto x_R - x_L$. This

[14] The concavity case contrasts, therefore, from the move to the center which can occur turnout is compulsory.

implies that $u_L = u_R$ for all pairs of policy positions, and so the turnout for a candidate relative to her opposition is not influenced by her policy choice. This gives her the opportunity to move toward her own preferred position. A natural case is one in which a candidate's preferred policy coincides with those of her faction. In this case, both candidates would move apart to the extremes of the policy space.

Corollary (to Proposition 10). *If voters' payoffs depend linearly on the implemented policy then the ratio of the candidates' turnout rates is invariant to local changes in their policy positions. Candidates who share the preferences of their factions' voters gain by shifting toward the extreme policy positions.*

But what if voters care about the identify of the winner as well as policy? Retaining a linear specification for the policy-loss function, suppose that

$$u_L = \bar{u}_L + \lambda_L(x_R - x_L) \tag{56}$$

$$u_R = \bar{u}_R + \lambda_R(x_R - x_L) \tag{57}$$

where $\max\{\bar{u}_L, \bar{u}_R\} > 0$. Here, for example, $\bar{u}_L$ is the net gain to the left-wing faction from a win by their candidate if policy positions are identical, while λ_L reflects the strength of left-wing policy concerns. If $\min\{\bar{u}_L, \bar{u}_R\} > 0$ then both candidates enjoy some factional loyalty. However, if $\bar{u}_R > 0 > \bar{u}_L$ or $\bar{u}_R < 0 < \bar{u}_L$ then one candidate enjoys a valence (or quality) advantage over her opponent: if both choose the same policy one candidate is preferred by all voters. (To maintain the feature that there is conflict between the two sides, I assume that the policy differences are sufficiently large to overturn any valence effects.)

Proposition 11. *Adopt the model specification of this session, and suppose that preferences satisfy $u_L = \bar{u}_L + \lambda_L(x_R - x_L)$ and $u_R = \bar{u}_R + \lambda_R(x_R - x_L)$. If candidate L has a valence disadvantage or if her faction has weak candidate preferences relative to their policy concerns, so that*

$$\frac{\bar{u}_L}{\lambda_L} < \frac{\bar{u}_R}{\lambda_R}, \tag{58}$$

then she gains electoral success by moving further to the left (u_L/u_R and her relative turnout rate are decreasing in x_L) but her opponent gains from a move toward the center (u_L/u_R is increasing in x_R).

A candidate with a valence or loyalty disadvantage (for the left-wing candidate, when $\bar{u}_L$ is relatively small or negative) pushes the voters' focus toward policy (and so levels the playing field between the candidates) by widening the gap between the policy choice (and so increasing $x_R - x_L$). The related incentive for the opponent means that both candidates face an incentive to shift to the left whenever candidate R has an advantage.[15] In a fuller policy-choice game (once again, this is not mapped out fully here) this naturally leads to an extremal position by the disadvantaged candidate (she gains both success and a more preferred policy by doing so) while the advantaged candidate faces a trade-off between electoral success and the desirability of her policy when she wins.

[15] This is consistent with predictions from the classic analysis of a Calvert-Wittman model (Wittman, 1977, 1983; Calvert, 1985) when one candidate has a valence advantage (Groseclose, 2001). Propositions 10–11 complement other reasons for the movement of candidate positions away from the Downs-Black median (for example, Aragones and Palfrey, 2002; Castanheira, 2003b; Kartik and McAfee, 2007; Carrillo and Castanheira, 2008; Herrera, Levine, and Martinelli, 2008).

8. Turnout in Larger Electorates

In this section I tackle two issues: the properties of turnout as the electorate size grows, and the use of approximations to the pivotal probabilities in my solution concept.

Turnout in Large Elections with Other-Regarding Preferences. For equilibria with incomplete turnout, the overall expected turnout rate is (Proposition 1)

$$\bar{t} = \frac{2\bar{p}(1-\bar{p})f(\bar{p})u}{cn}. \tag{59}$$

Other things equal, this falls with the electorate size, and so it is tempting to conclude that the turnout rate will be low in large electorates; this is at the heart of the turnout paradox. For higher turnout rates in large electorates, many authors have argued that the odds of influence are stacked against a voter.[16] Extending the scenario described in Proposition 4, substantial turnout in a population of 1,000,000 requires a voter to participate for odds of 1-in-25,000. If a voter cares only about his narrow material self-interest and if it costs $5 to vote then the success of favored candidate must make a difference of $125,000 to his life. Viewed narrowly (as, for example, the effect of a fiscal policy change on a private individual) this instrumental benefit does seem rather large.

However, rational choices do not require a voter to be selfish. For example, a voter may wish to elect the best candidate for his society. Moreover, in larger electorates the issues at stake are different, the pattern of preferences may differ, and voters may have better information. Focusing on the first point, in a larger electorate the stakes are higher.[17] At a basic level, this is simply because of the number of people who are affected by the outcome. If a voter has other-regarding social preferences (he cares about others, whether altruistically or paternalistically) then the weight of the election outcome will increase with n.

Adopting this argument formally, suppose that the instrumental payoff is contingent on the electorate size, and denoted as u_n. Suppose that this is linearly increasing in n, so that

$$u_n = \bar{u} + bn. \tag{60}$$

The payoff $\bar{u}$ is the private implication of the outcome, whereas b is the per-person impact on others from the perspective of an individual voter. The parameter b reflects the voter's social preferences. (Many critiques of instrumental theories implicitly assume that $b = 0$, without arguing why that should be the case.) The expected turnout rate is

$$\bar{t}_n = \frac{2\bar{p}(1-\bar{p})f(\bar{p})(\bar{u}+bn)}{cn} \quad \Rightarrow \quad \lim_{n\to\infty} \bar{t}_n = \frac{2\bar{p}(1-\bar{p})f(\bar{p})b}{c}. \tag{61}$$

A n grows the turnout rate does not vanish; it converges to a non-zero limit. The other comparative-static results reported in Proposition 1 apply to this limit.

[16] For example, Owen and Grofman (1984, p. 322) claimed that if a voter enjoys "only a one-in-21,000 chance of affecting the election" then he would be "best off staying home."

[17] My review of the literature (Section 9) reports that this suggested has been made elsewhere, most notably by Edlin, Gelman, and Kaplan (2007), and was developed in independent work by Evren (2012).

One comparative static reported in Proposition 3 is that turnout increases with the precision of beliefs. This precision is related to the extent of pre-election information. In a larger electorate, opinion polling is more common. As n rises, we might expect $\text{var}[p]$ to fall, and so $f(\bar{p})$ to rise. This can lead to an increasing relationship between electorate size and turnout.

Proposition 12 (Equilibrium with Social Preferences)**.** *If voters have social preferences of the form $u_n = \bar{u} + bn$, then the turnout rate converges to a non-zero limit as the electorate grows. This limit is determined by the extent of voters' other-regarding preferences relative to the cost of voting. If the size of pre-election polls (corresponding to s in the Beta specification) increases with the election's size and importance, then (at least for large n) the turnout rate will increase with the electorate size.*

Expanding the "small city" scenario (Proposition 4) once more, to explain substantial turnout (50%) in a larger population of 10,000,000 I need voters who are willing to show up for the longer odds of 1-in-250,000. If voting costs \$5, then this places an implicit value of \$1,250,000 for changing the outcome. This is well beyond the private consequences of an election. However, a tiny element of social preferences is enough to get things to work. Revisiting a scenario described in my introductory remarks, suppose that a voter believes that the right winner helps the typical citizen by \$250 per year over a five year term; a modest impact of \$1,250. If this voter's concern for others is only 0.0001 (one hundredth of one per cent) then the implicit value of the election to him is indeed \$1,250,000, and he would participate for the stated chance of influencing the outcome. Social preferences scale up, and so the same other-regarding preferences work with a population of 5,000,000 (an example from the introduction) as well as a population of 100,000,000 (a national election in a large country).

Asymptotic Voting Equilibria. Adopting the specification of equation (60) also helps me to bolster the solution concept. In a voting equilibrium voters act optimally given that they use the approximations from Lemma 3. These approximations improve as n grows; but of course, as n grows the size of the pivotal probabilities shrinks, and so (when payoffs do not depend on n) the turnout rate does so too. Thus, to use limiting values properly, I need to ensure that the equilibrium turnout rates do not vanish. Allowing payoffs to increase linearly with n (which is arguably more general than fixing u) is a way to do this. Doing so, a pair of turnout probabilities can act as an appropriate solution for all larger electorates.

To see this idea in action, here I adopt a variant of the equilibrium concept used by Dewan and Myatt (2007) and by Myatt (2007, 2015). This concept is defined relative to a sequence of voting games. Separate equilibrium turnout probabilities are not specified for each electorate size. Instead, I take a pair of turnout probabilities and ask whether a voter can do more than ε better in large electorates; thus I look for a kind of ε-equilibrium. I then seek the turnout probabilities for which ε can be made arbitrarily small in large electorates.

Definition (Solution Concept)**.** *The turnout probabilities t_L and t_R are an* asymptotic voting equilibrium *if a voter's gain from switching his turnout decision converges to zero as $n \to \infty$.*

If the instrumental payoff increases linearly with n (or more generally if $\lim_{n\to\infty}(u_n/n) = b$) then the asymptotic voting equilibrium solution concept makes a unique prediction.

Proposition 13 (Asymptotic Voting Equilibrium with Social Preferences)**.** *If voters have social preferences of the form $u_n = \bar{u} + bn$ and c is not too small then there is a unique asymptotic voting equilibrium, with turnout probabilities*

$$t_L = \frac{\bar{p} f(\bar{p}) b}{\bar{a} c} \quad \textit{and} \quad t_R = \frac{(1-\bar{p}) f(\bar{p}) b}{\bar{a} c}. \tag{62}$$

Obviously, t_L and t_R are the limiting values of the solutions already obtained in this paper. The comparative-static results already derived hold here too. Furthermore, it is straightforward to extend the social preferences specification to other environments. For instance, in the environment with uniformly distributed voting costs the expected turnout rate is

$$\bar{t} = \sqrt{\bar{a} b \sqrt{\bar{p}(1-\bar{p})} f\left(\frac{\sqrt{\bar{p}}}{\sqrt{\bar{p}} + \sqrt{1-\bar{p}}}\right)}. \tag{63}$$

9. RELATED LITERATURE

The turnout literature has been surveyed by Blais (2000), Dhillon and Peralta (2002), Feddersen (2004), Dowding (2005), Geys (2006a,b), and many others. The early literature viewed the influence of a vote as too small to generate turnout, and so emphasized factors such as civic duty (Riker and Ordeshook, 1968); a brief renaissance conjectured that game-theoretic models could predict reasonable turnout (Palfrey and Rosenthal, 1983; Ledyard, 1981, 1984); and, finally, it was recognized that such models relied on a knife-edge property (Palfrey and Rosenthal, 1985). The status quo is that (Feddersen and Sandroni, 2006b, p. 1271) "there is not a canonical rational choice model of voting in elections with costs to vote."

Most models of turnout restrict to voter types that are independent draws from a known distribution. This is also true for models of strategic voting (Cox, 1984, 1994; Palfrey, 1989; Fey, 1997), of the signaling motive for voting (Meirowitz and Shotts, 2009), and of the welfare performance of voluntary voting (Campbell, 1999; Börgers, 2004; Krasa and Polborn, 2009; Taylor and Yildirim, 2010b; Krishna and Morgan, 2011, 2014).[18] There is no aggregate uncertainty, and so the support for each candidate is essentially known in a large electorate.

Such "independent type" models have other difficult features. Away from knife-edge cases the probability of a tie decays exponentially as the electorate grows. This suggests that the influence of an individual is negligible. However, Good and Mayer (1975) elegantly demonstrated that aggregate uncertainty changes this claim. If n voters participate and if a voting probabiltiy p is drawn from a density $f(\cdot)$, then the tie probability approximately $f(\frac{1}{2})/2n$. This is inversely proportional (and so not exponentially related) to the electorate size. This result, which was also reported by Chamberlain and Rothschild (1981), "has largely been ignored, forgotten or unknown by most who have needed to calculate the value of [the probability of being decisive]" (Fischer, 1999). However, it is not directly applicable when some

[18]Voting power indices (Penrose, 1946; Banzhaf, 1965) also rely on independent-type specifications (Gelman, Katz, and Tuerlinckx, 2002; Gelman, Katz, and Bafumi, 2004; Kaniovski, 2007). In contrast, some form of aggregate uncertainty is usually present in common-value jury models (for example, Austen-Smith and Banks, 1996; Feddersen and Pesendorfer, 1996, 1997, 1998; Austen-Smith and Feddersen, 2006; Mandler, 2012; McMurray, 2013; Bhattacharya, 2013; Bouton, Llorente-Saguer, and Malherbe, 2015) in which a voter conditions-on-being-pivotal to ascertain his own preferred option.

voters abstain. (Good and Mayer (1975) considered voters with only two options. A model with voluntary turnout must also allow for a third option—namely, abstention.) A small technical contribution here is to extend the Good-Mayer result beyond the binary-option setting, so enabling the analysis of voluntary turnout with aggregate uncertainty.

Notably, Edlin, Gelman, and Kaplan (2007) recognized that a vote's influence should be "roughly proportional to $1/n$" and suggested that other-regarding concerns might support substantial turnout.[19] Such social preferences are related to the notion (Jankowski, 2002, 2007) that a costly vote is "a lottery ticket to help the poor." Edlin, Gelman, and Kaplan (2007, p. 293) persuasively argued that while "the probability that a vote is decisive is small" nevertheless "the social benefits at stake in the election are large, and so the expected utility benefit of voting to an individual with social preferences can be significant." However, they did not specify a model or offer proofs. A contribution here (in Section 8) is to establish that aggregate uncertainty and mild social preferences can jointly explain substantial turnout. Importantly, this aspect overlaps with independent work by Evren (2012) who specified aggregate uncertainty over the proportion of voters with altruistic preferences.

In equilibrium, the probability of a pivotal vote exceeds the "$1/N$" from Tullock's (1967) classic reasoning. For example, it is approximately $40/N$ in the scenario of Proposition 4 and so is closer to empirical estimates (Gelman, King, and Boscardin, 1998; Mulligan and Hunter, 2003). This is because of the underdog effect: if a candidate's turnout is inversely related to her expected popularity then the likelihood of a close race is maximized. Underdog effects have been studied in recent work on the welfare properties of systems with voluntary participation (Goeree and Großer, 2007; Krasa and Polborn, 2009; Taylor and Yildirim, 2010a,b; Krishna and Morgan, 2011, 2014; Herrera, Morelli, and Palfrey, 2014). For example, Taylor and Yildirim (2010a) used the fact that different types are interested in different close-call events: a left-wing voter is interested in situations in which the left-wing candidate is one vote behind (an extra vote creates a tie, and so a possible left-wing win) while a right-wing voter considers outcomes in which the right-wing candidate is one vote behind. (Both types of voters are interested in an exact tie.) In equilibrium, both types perceive the same probability of being pivotal and so the one-vote-behind events must be equally likely. This happens when the turnout rates reverse any popularity-derived advantage for one of the candidates. However, this mechanism does not work in the presence of aggregate uncertainty because the one-vote-behind outcomes are equally likely in a large electorate (Lemma 2).[20]

[19]This idea also appears in other work, such as articles by Fowler and Kam (2007), Loewen (2010), and Dawes, Loewen, and Fowler (2011). Past empirical work has reported evidence that voters incorporate so-called sociotropic (society level) factors (Kinder and Kiewiet, 1981; MacKuen, Erikson, and Stimson, 1992; Clarke and Stewart, 1994; Mutz and Mondak, 1997); participation in elections is associated with measures of social cooperation, such as jury service and census response (Knack, 1992b,a; Knack and Kropf, 1998); and experimental researchers have reported an association between the self-reported electoral turnout behavior of subjects and the extent of altruistic allocations in a dictator game (Fowler, 2006). Notice that the social preferences considered here are derived from a voter's anticipated instrumental effect on the electoral outcome, and so differ from the addition of a civic duty term to a voter's payoff (Riker and Ordeshook, 1968; Goldfarb and Sigelman, 2010), from the use of an ethical voter model (Feddersen and Sandroni, 2006a,b; Feddersen, Gailmard, and Sandroni, 2009), or from the pressure of social norms (Gerber, Green, and Larimer, 2008).

[20]Some of the papers discussed here do incorporate some element of aggregate uncertainty. Taylor and Yildirim (2010a) did so, but restricted to symmetric prior beliefs so that no underdog exists. Goeree and Großer

The underdog effect is resurrected because of introspection: a voter uses his type to update his beliefs about p. When evaluated at the mean $\bar{p} \equiv \mathrm{E}[p]$ an application of Bayes' rule confirms that $f(\bar{p} \mid L) = f(\bar{p} \mid R) = f(\bar{p})$; that is, when thinking about the likelihood that the underlying division of support is equal to its expectation, a voter's beliefs do not shift when he conditions on his type. What this means is that voters' beliefs coincide when they worry about the likelihood that $p = \bar{p}$; and they are concerned about this only when $p = \bar{p}$ results in a close-run race. For this to be so, the turnout rates amongst different factions must be inversely related to the corresponding candidates' expected popularities.

I have noted here that the literature contains relatively few models with aggregate uncertainty over voters' preferences. There has, however, been the development of models with an uncertain electorate size. This research builds upon work by Myerson (1998a,b, 2000, 2002) in which the (large) number of players is a Poisson variable. The applications have included elections with vote-share-contingent policies (Castanheira, 2003a), approval voting (Núñez, 2010), the evaluation of scoring rules (Goertz and Maniquet, 2011), voting in runoff elections (Martinelli, 2002; Bouton, 2013; Bouton and Gratton, 2015), behavior in multi-candidate plurality-rule elections (Bouton and Castanheira, 2012), and of course turnout itself (Herrera, Morelli, and Palfrey, 2014). Myerson (1998a, p. 112) explained that "Poisson games ... have some very convenient technical properties." The numbers of players associated with each action (votes for L, votes for R, and abstentions) are independent random variables. This property helps in the calculation of pivotal probabilities. In this paper I suggest that it is aggregate uncertainty over voters' preferences that matters. In particular, I allow not only for aggregate uncertainty over the popularities of the candidates (the probability p) but also over the effective electorate size (by supposing that a voter is only available to vote with probability a, where a is uncertain). The uncertainty over the electorate size is unimportant for the results, but the uncertainty over voters' preferences is crucial.

In summary, most researchers have not specified aggregate uncertainty in their models and yet such uncertainty is the critical ingredient (Good and Mayer, 1975). There are some exceptions; recent models of strategic and protest voting have included aggregate uncertainty (Dewan and Myatt, 2007; Myatt, 2007, 2015; Myatt and Smith, 2014). Hence, relative to most existing literature, this paper brings aggregate uncertainty (over both candidates' popularity, and the available electorate) to a model of voter turnout. It also develops full comparative-static results, investigates the implications for office-seeking candidates' policy choices, and evaluates the social performance of elections with voluntary turnout.

(2007) allowed two-point support for p focused either on a symmetric prior or on a situation without aggregate uncertainty. An asymmetric prior (so that an underdog exists) with aggregate uncertainty was only considered with two voters. Krasa and Polborn (2009) specified uncertainty about a candidate's popularity, but that uncertainty is resolved before voters act. Ghosal and Lockwood (2009) considered a model where a voter's private preference type is independently drawn from a known distribution, but where there is a common-value element to the payoff from each candidate about which voters' observe informative signals. As in the model of Börgers (2004), the two private preference types are equally likely. A general theme throughout this strand of literature is this: either there is no aggregate uncertainty, or there is a symmetric specification for beliefs about candidates' popularities. In fact, Börgers (2004) imposed both independent types and symmetry.

10. THE TURNOUT PARADOX

In an oft-quoted question based on a statement of Fiorina (1989), Grofman (1993) asked: "is turnout the paradox that ate rational choice theory?" More recently, Blais (2000, p. 2) supported the wasted-vote argument, even in a moderately sized electorate: "with 70,000 voters, even in a close race the chance that both candidates will get exactly the same number of votes is extremely small." While acknowledging that the cost of voting is small, he reasoned that "the expected benefit is bound to be smaller for just about everyone because of the tiny probability of casting a decisive vote" and so the calculus-of-voting model (Downs, 1957; Riker and Ordeshook, 1968) "does not seem to work."[21] The turnout paradox has been used by some (Green and Shapiro, 1994, notably) to argue forcefully against the use of rational-choice methods from economics in political scientific settings.

A narrowly self-interested and instrumentally motivated citizen might not turn out for the odds discussed here. From this, the Green and Shapiro (1994) argument invites researchers to abandon rational-choice methods. In essence, their critique requires voters to be dumb in order to cling on to the assumption of very narrow selfishness. However, allowing smart voters to be very mildly (as little as 0.01%) socially motivated can explain significant turnout; rational choice does not require narrow selfishness. In the presence of aggregate uncertainty (a reasonable property) a rational-choice model is compatible with realistic turnout rates in moderately sized electorates; and minimal other-regarding preferences are enough to ensure compatibility in larger electorates. In summary: I join Edlin, Gelman, and Kaplan (2007), Evren (2012) and others in concluding that there is no paradox.

Aldrich (1993, p. 246) observed that "turning out to vote is the most common act citizens take in a democracy" and yet "it is not well understood." To provide this understanding, the theory presented here uses a simple model with largely standard features, but develops fully the (perhaps neglected) insights of Good and Mayer (1975). The conclusions are that significant turnout is consistent with goal-oriented voters under reasonable parameter considerations; the underdog effect offsets (although not necessarily completely) the advantage of a perceived leading candidate via greater turnout for her competitor; and the precision of voters' beliefs, as well as other more familiar factors, has a predictable impact on turnout in elections. The results on candidate positioning show how policies may be chosen when success is determined by relative turnout rather than chasing the median voter. Furthermore, the social desirability of elections with voluntary turnout depends upon the size of the underdog effect: the election biases against a perceived leader, and so cannot be relied upon to select a utilitarian winner. These additional results show that the model mcan be used as a component in future models of democratic systems.

[21]As noted in my discussion of the literature, this view is well-established elsewhere. For instance, Barzel and Silberberg (1973) looked back to Arrow (1969, p. 61) who said that it is "hard to explain ... why an individual votes at all in a large election, since the probability that his vote will be decisive is so negligible" while Goodin and Roberts (1975, p. 926) advised that "the politically rational thing to do is to conserve on shoe leather." Many other have accepted this conclusion; a survey by Dhillon and Peralta (2002) led by quoting Aldrich (1997), who said that "the rationality of voting is the Achilles' heel of rational choice theory in political science."

Proof of Lemma 1. I prove the statement that

$$\lim_{n\to\infty} \max_{b\in\Delta_n^\dagger} \left| \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \Pr[b] - h\left(\frac{b}{n}\right) \right| = 0$$

from which the lemma follows. Using equation (4) from the text,

$$\frac{\Gamma(n+m+1)}{\Gamma(n+1)} \Pr[b] = \int_\Delta \frac{\Gamma(n+m+1)}{\prod_{i=0}^m \Gamma(b_i+1)} \left[\prod_{i=0}^m v_i^{b_i}\right] h(v)\, dv$$

$$= h\left(\frac{b}{n}\right) \int_\Delta \underbrace{\frac{\Gamma(n+m+1)}{\prod_{i=0}^m \Gamma(b_i+1)} \left[\prod_{i=0}^m v_i^{b_i}\right]}_{\text{Dirichlet density}} dv + \int_\Delta \frac{\Gamma(n+m+1)}{\prod_{i=0}^m \Gamma(b_i+1)} \left[\prod_{i=0}^m v_i^{b_i}\right] \left[h(v) - h\left(\frac{b}{n}\right)\right] dv$$

$$= h\left(\frac{b}{n}\right) + \underbrace{\int_\Delta \frac{\Gamma(n+m+1)}{\prod_{i=0}^m \Gamma(b_i+1)} \left[\prod_{i=0}^m v_i^{b_i}\right] \left[h(v) - h\left(\frac{b}{n}\right)\right] dv}_{\text{vanishes uniformly}},$$

The remainder of the proof proceeds by finding a uniform lower bound for the size of the final term, and therefore showing (as indicated) that it declines uniformly to zero.[22]

For $\gamma \in \Delta$ define $\Delta_\varepsilon^\gamma$ as an ε neighborhood of γ. That is: $\Delta_\varepsilon^\gamma = \{v \in \Delta \,|\, \max|v_i - \gamma_i| \le \varepsilon\}$. $h(\cdot)$ is a continuous density with bounded derivatives and so there is some positive D, which does not depend on γ, such that $|h(v) - h(\gamma)| \le D\varepsilon$ for all $v \in \Delta_\varepsilon^\gamma$. Furthermore, $\bar{h} = \max_{v\in\Delta} h(v)$ bounds the difference $|h(v) - h(\gamma)|$ for all $v \notin \Delta_\varepsilon^\gamma$. Hence

$$\left| \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \Pr[b] - h\left(\frac{b}{n}\right) \right| \le \int_\Delta \frac{\Gamma(n+m+1)}{\prod_{i=0}^m \Gamma(b_i+1)} \left[\prod_{i=0}^m v_i^{b_i}\right] \left| h(v) - h\left(\frac{b}{n}\right) \right| dv$$

$$\le D\varepsilon + \bar{h} \int_{\Delta/\Delta_\varepsilon^{b/n}} \frac{\Gamma(n+m+1)}{\prod_{i=0}^m \Gamma(b_i+1)} \left[\prod_{i=0}^m v_i^{b_i}\right] dv$$

$$= D\varepsilon + \bar{h} \int_{\Delta/\Delta_\varepsilon^{b/n}} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \Pr[b \,|\, v]\, dv.$$

For $v \notin \Delta_\varepsilon^{b/n}$ there is some i such that $v_i < (b_i/n) - \varepsilon$ or $v_i > (b_i/n) + \varepsilon$. Consider the latter case (the former case is similar) and note that b_i is a binomial with parameters v_i and n. Hence

$$\Pr[b \,|\, v] \le \Pr\left[\frac{b_i}{n} \le v_i - \varepsilon\right] \le e^{-2n\varepsilon^2},$$

where the second inequality is well known (Hoeffding, 1963). It follows that

$$\left| \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \Pr[b] - h\left(\frac{b}{n}\right) \right| \le D\varepsilon + \bar{h} \int_{\Delta/\Delta_\varepsilon^{b/n}} \frac{\Gamma(n+m+1)}{\Gamma(n+1)} \Pr[b \,|\, v]\, dv$$

$$\le D\varepsilon + \bar{h} e^{-2n\varepsilon^2} \frac{\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+1)} \int_{\Delta/\Delta_\varepsilon^{b/n}} \Gamma(m+1)\, dv \le D\varepsilon + \bar{h} e^{-2n\varepsilon^2} \frac{(n+m)^m}{\Gamma(m+1)}$$

where the final inequality is obtained by expanding the range of integration to include all of Δ, by recognizing once more a Dirichlet density, and by bounding above the ratio of Gamma

[22] This proof uses a shorter version of the approach used in the supplement to Hummel (2012). His Lemma 3 corresponds to the case $m = 2$, whereas the Good-Mayer-Chamberlain-Rothschild result applies to $m = 1$.

functions. This bound holds for any ε. Take, for example, $\varepsilon = n^{-1/4}$. For this choice of ε

$$\left|\frac{\Gamma(n+m+1)}{\Gamma(n+1)}\Pr[b] - h\left(\frac{b}{n}\right)\right| \leq \frac{D}{n^{1/4}}\varepsilon + \bar{h}e^{-2\sqrt{n}}\frac{(n+m)^m}{\Gamma(m+1)}.$$

Noting that the exponential term dominates the polynomial term, this bound (which is independent of b, hence yielding uniform convergence) vanishes as $n \to 0$. ☐

Proof of Lemma 2. From (6) in the text, $n\Pr[b_L = b_R] = \frac{1}{n}\sum_{z=0}^{\lfloor n/2\rfloor} n^2 \Pr[b_L = b_R = z \mid i]$. Using Lemma 1, $n^2\Pr[b_L = b_R = z]$ converges uniformly. That is,

$$\lim_{n\to\infty} \max_{z\in\{0,1,\ldots,\lfloor n/2\rfloor\}} \left| n^2 \Pr[b_L = b_R = z] - h\left(\frac{n-2z}{n}, \frac{z}{n}, \frac{z}{n} \mid i\right)\right| = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} n\Pr[b_L = b_R \mid i] = \lim_{n\to\infty} \frac{1}{n}\sum_{z=0}^{\lfloor n/2\rfloor} h\left(\frac{n-2z}{n}, \frac{z}{n}, \frac{z}{n} \mid i\right),$$

where this holds so long as the right-hand side limit is well-defined. As noted in the text, the right-hand side defines a Riemann integral of $h(1-2x, x, x)$ over the range $[0, 1/2]$, and so converges to $\int_0^{1/2} h(1-2x, x, x)\, dx$ as $n \to \infty$. Similar calculations apply to near ties. ☐

Proof of Lemma 3. Substitute the conditional densities from (13) into (12). ☐

Proof of Lemma 4. As noted in the main text, (17) shows that $\Pr^\dagger[\text{Pivotal} \mid L] = \Pr^\dagger[\text{Pivotal} \mid R]$ if and only if $\bar{p} = p^\star = t_L/(t_L + t_R)$. ☐

Proof of Lemma 5. $c < \frac{v}{2}$ and so there must be positive turnout from at least one voter type. This implies that $\max\{t_L, t_R\} > 0$, and so $p^\star$ (from Lemma 3) is defined. If $t_R \geq t_L$ then $t_R > 0$ and so $\Pr^\dagger[\text{pivotal} \mid R] > 0$. $t_R \geq t_L$ implies that $p^\star \leq \frac{1}{2}$ and so, given that $\bar{p} > \frac{1}{2}$, the expressions in (17) ensure that $\Pr^\dagger[\text{pivotal} \mid R] < \Pr^\dagger[\text{pivotal} \mid L]$. This means there is a strict incentive for supporters of L to participate, and so $t_L = 1$. Hence, $t_R \geq t_L$ can only hold if $t_L = t_R = 1$; so, it cannot be the case that $t_R > t_L$. The final claim follows directly. ☐

Proof of Proposition 1. Using equation (18) and equating to (c/u) readily yields $\bar{t}$. Combining this with the fact that $t_L/(t_L + t_R) = \bar{p}$ from Lemma 4 yields the expressions for t_L and t_R in equation (19). The comparative-static claims follow by inspection. The proposition says that "if (c/u) is not too small." This is so that the solutions satisfy $\max\{t_L, t_R\} < 1$; the relevant inequality is stated in the main text as equation (20). The uniqueness of this incomplete-turnout equilibrium for (c/u) satisfying this inequality is established by Proposition 2. ☐

Proof of Proposition 2. From Lemma 5, the possible equilibria are: (i) complete turnout from both factions, so that $t_L = t_R = 1$; (ii) complete turnout for the underdog, so that $t_R < t_L = 1$; and (iii) incomplete turnout from both factions, so that $t_L < t_R < 1$.

Case (i). If $t_L = t_R = 1$ then $p^\star = \frac{1}{2}$. Given that $\bar{p} \geq \frac{1}{2}$, this means that $\Pr^\dagger[\text{pivotal} \mid L] \geq \Pr^\dagger[\text{pivotal} \mid R]$. Hence, a necessary and sufficient condition for this to be an equilibrium is

$$\Pr^\dagger[\text{pivotal} \mid R] \geq \frac{c}{v} \quad \Leftrightarrow \quad f(p^\star)(p^\star)^2 \geq \frac{c\bar{a}n\bar{p}}{v} \quad \text{where} \quad p^\star = \frac{1}{2}. \tag{i}$$

Case (ii). If $t_R < t_L = 1$ then $p^\star = 1/(1+t_R) > \frac{1}{2}$. For this to be an equilibrium the supporters of R must be indifferent and so $\Pr^\dagger[\text{pivotal} \mid R] = (c/v)$. The supporters of L must also be willing to turn out, which is so if and only if $\Pr^\dagger[\text{pivotal} \mid L] \geq \Pr^\dagger[\text{pivotal} \mid R] \Leftrightarrow p^\star \leq \bar{p}$. There is an equilibrium of this kind if and only if there is a $p^\star$ satisfying

$$f(p^\star)(p^\star)^2 = \frac{c\bar{a}n\bar{p}}{v} \quad \text{where} \quad \frac{1}{2} < p^\star \leq \bar{p}. \tag{ii}$$

Case (iii). If $t_R < t_L < 1$ then $p^\star = \bar{p}$. The turnout rates for such an equilibrium are reported in Proposition 1. As noted in the text, there is an equilibrium of this kind only when inequality (20) is satisfied, which holds if and only if

$$f(p^\star)(p^\star)^2 < \frac{c\bar{a}n\bar{p}}{v} \quad \text{where} \quad p^\star = \bar{p}. \tag{iii}$$

Notice that $f(p)p^2$ is a continuous in p and achieves a maximum on the compact interval $[(1/2), \bar{p}]$. If $v \max_{p \in [(1/2), \bar{p}]}[f(p)p^2] < c\bar{a}n\bar{p}$ neither (i) nor (ii) are satisfied but (iii) is, and so there is a unique equilibrium with incomplete turnout from both factions. This proves the uniqueness claim of Proposition 1; note that this does not require monotonicity of $f(p)p^2$.

Now, however, suppose that $f(p)p^2$ is strictly increasing over the relevant interval (as is stipulated in the proposition) and so is maximized at $p = \bar{p}$ and minimized at $p = \frac{1}{2}$. Thus, if $v[f(\bar{p})\bar{p}^2] < c\bar{a}n\bar{p}$ then (iii) holds but (i) and (ii) do not. However, if $v[f(\bar{p})\bar{p}^2] \geq c\bar{a}n\bar{p}$ then (iii) fails. If $v[f(1/2)(1/2)^2] \geq c\bar{a}n\bar{p}$ then (i) holds, but (ii) must fail, and so there is a unique equilibrium with complete turnout from everyone. The remaining case is when $v[f(\bar{p})\bar{p}^2] \geq c\bar{a}n\bar{p} > v[f(1/2)(1/2)^2]$. Neither (i) nor (iii) hold. However, $f(p)p^2$ is continuous and strictly monotonic, and so (ii) has a unique solution in the relevant interval; hence there is a unique equilibrium involving complete turnout for the underdog. □

Proof of Proposition 3. The expression for $\bar{t}$ is obtained by substituting in the normal density for $f(\bar{p})$. The comparative-static claim holds by inspection. □

Proof of Proposition 4. This follows from the calculations performed in the main text. □

Proof of Proposition 5. Substituting in the expressions for $\Pr^\dagger[\text{pivotal} \mid L]$ and $\Pr^\dagger[\text{pivotal} \mid R]$, the equilibrium conditions from equation (31) become

$$\frac{f(p^\star)(1-p^\star)}{\bar{a}n(t_L+t_R)(1-\bar{p})} = \frac{c_L}{u} \quad \text{and} \quad \frac{f(p^\star)p^\star}{\bar{a}n(t_L+t_R)\bar{p}} = \frac{c_R}{u}.$$

Taking the ratio of these two equations,

$$\frac{p^\star(1-\bar{p})}{\bar{p}(1-p^\star)} = \frac{c_R}{c_L},$$

which is re-arranged to yield equation (32). Notice that $p^\star = \bar{p} \Leftrightarrow c_L = c_R$. Solving for the equilibrium turnouts is straightforward; simple algebra confirms the solutions stated in the proposition, and $\bar{p}t_R > (1-\bar{p})t_L \Leftrightarrow c_R < c_L$ by inspection. The turnout rates depend on $f(p^\star)$. Away from the mean (for $p^\star \neq \bar{p}$) the density of the normal is non-monotonic (first increasing, and then decreasing) as the variance falls (the precision rises). □

Proof of Proposition 6. The first claim follows from equation (35). Re-arranging this equation yields $p^\star$. The solutions for t_L and t_R are obtained by solving the equilibrium conditions after setting $C(t_L) = t_L$ and $C(t_R) = t_R$. The remaining claims follow by inspection. □

Proof of Proposition 7. The claims follow from equation (40). □

Proof of Proposition 8. The claims follow from the argument in the text. □

Proof of Proposition 9. For $t_L > t_R$ note that the concavity of $C(\cdot)$ implies

$$\frac{C(t_L)}{C(t_R)} - \frac{t_L}{t_R} = \frac{C(t_L) - C(t_R)}{C(t_R)} - \frac{t_L - t_R}{t_R} < \frac{(t_L - t_R)C'(t_R)}{t_R C'(t_R)} - \frac{t_L - t_R}{t_R} = 0. \tag{64}$$

The claims now follow from the discussion in the text. □

Proof of Proposition 10. Differentiate the expression for u_L/u_R in equation (55). □

Proof of Proposition 11. Take the ratio of u_L and u_R from equations (56) and (57) and differentiate with respect to x_R and x_L repectively. □

Proof of Proposition 12. The claims follow from equation (61). □

Proof of Proposition 13. Fixing t_L and t_R, a voter's gain or loss from switching his participation decision converges to zero as $n \to \infty$ if and only if the expected benefit from voting converges to c. For these turnout probabilities to yield an asymptotic voting equilibrium,

$$\lim_{n\to\infty} (\bar{u} + bn) \Pr[\text{pivotal} \mid L] = \lim_{n\to\infty} (\bar{u} + bn) \Pr[\text{pivotal} \mid R] = c.$$

Now, using Lemma 3,

$$(\bar{u} + bn) \Pr[\text{pivotal} \mid L] = n \Pr[\text{pivotal} \mid L] \times \frac{\bar{u} + bn}{n} \to \frac{bf(p^\star)}{\bar{a}(t_L + t_R)} \frac{1 - p^\star}{1 - \bar{p}} \quad \text{as} \quad n \to \infty,$$

with a similar calculation holding for a supporter of candidate R. Setting these limits equal to c yields the expressions for t_L and t_R given in the statement of the proposition. □

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