Finance in a Time of Disruptive Growth

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Abstract

Increased arrival of new technologies and displacement of old technologies leads to increased heterogeneity of investment income across investors. Investors who find themselves with high exposures to successful new firms and entrepreneurs win a disproportionate share of profits, while those who hold lower exposures may end up with a lower fraction of aggregate investment income. Using a novel data set on the wealth of ultra-high net worth individuals, we document the rapid ascension to wealth by self-made businesspeople in the last decades. We then build a model that encompasses lack of risk sharing between existing investors and also between existing investors and arriving entrepreneurs to study the implications of redistributive growth for wealth dynamics and portfolio choices of investors in a dynamic general equilibrium model. We show that "alternative asset classes" as diverse as commercial real estate, commodities, and private equity offer hedging and diversification benefits in a world of increased displacement, and therefore experience inflows. The same applies to zero-net supply risk free investments, leading to a drop in the risk free rate. The output share of the financial industry increases, whereas (irreversible) investment in specialized equipment and structures may not, despite the lower required rates of return. Surprisingly, there is a positive gap between the expected returns of investments in new versus existing firms, despite the diversification benefits offered by the former.

1 Introduction

The last few decades have seen a change in the investment landscape. Real interest rates have experienced a protracted decline. At the same time alternative asset classes that seem to have very little to do with each other (private equity, venture capital, commercial real estate, commodities, etc.) have attracted increased portfolio allocations. We link these trends to an increased incidence of growth that is "disruptive" — or, more appropriately, redistributive: Growth leads new firms to capture a larger portion of profits and market capitalization, and routinely at the expense of old firms which get displaced. The creation of these companies benefits investors asymmetrically: The benefits accrue predominantly to their creators (and more generally investors holding large fractions of their equity), but not necessarily to an investor simply holding the market portfolio of public companies.

If the dispersion of wealth growth across investors increases, then investors have an incentive to increase allocations towards asset classes that are less affected by displacement risk. Our model identifies private equities, real assets and risk-free assets as the asset classes that fit that requirement, thus explaining their popularity in recent years.

Specifically, we develop a dynamic general equilibrium model with the following features. The production of a final good requires labor and intermediate products. New lines of (intermediate) products raise aggregate production, but also displace the demand and hence the profits for old intermediate products. The ownership rights to the production of the new product lines are allocated either to existing publicly traded firms or to newly arriving agents. The allocation of blueprints to these new agents is highly asymmetric. Few of these agents end up with quite profitable product lines, while the rest end up with worthless allocations. As a result, these newly arriving agents are eager to share that risk with investors, by offering a fraction of their company's shares for sale. This transaction is facilitated by intermediaries who purchase a portfolio of these privately held shares. However, this diversification is costly: In particular the costs of diversifying across all shares would be prohibitively costly, and thus the intermediary invests only in a subset of these firms.

Thus there are two dimensions that cause risk sharing to be imperfectly shared: a) An inter-cohort dimension (across newly arriving agents and existing investors), which depends on the fraction of shares that are retained by the newly arriving agents. b) An intra-cohort dimension (among existing investors), which depends on the correlation between an intermediary's portfolio and the market value of all blueprints accruing to firms outside publicly traded equities. Indeed, the model nests the perfect risk sharing limit, the Constantinides-Duffie (1996) model, and the OLG model of Garleanu, Kogan, and Panageas (2012) as special parametric cases.

We provide explicit closed-form solutions of the model and show the following results.

First, if risk sharing is close to perfect, then an increased arrival of new technologies is "good news" for the marginal investor, since new technologies are good news for aggregate output and market capitalization. However, if either intra- or inter-cohort risk sharing fails, then increased arrival of new technologies is perceived as risky by the representative investor, who might end up losing from the new technologies.

Second, a surprising result of the model is that the equilibrium returns of privately held firms must exceed the expected returns of publicly traded firms, even though they offer hedging opportunities against displacement risk. The reason is that in a world of imperfect risk sharing the diversifiable risk of private investments commands risk compensation, even if the investor's portfolio places a small weight in such investments. Given the notorious difficulties of measuring the returns of such investments, our model provides a simple theoretical explanation for the fact that privately held firm investments seem to produce high gross average returns in the data

Third, an acceleration in displacement or dispersion of innovation gains across investors increases the incentives to diversify out of public equities. The natural targets are risk free assets, private equities, or real assets. Real assets benefit from increased arrival of new firms since all firms (new and old) need assets such as real estate and commodities. Private equity benefits because it helps offset the displacement risk of public equities; as a result the size of the financial industry that facilitates that transfer expands. The demand for the risk-free asset increases due to precautionary savings incentives – and since it is in zero net supply, this leads to a decline in the risk free rate.

Fourth, an acceleration in displacement or dispersion in innovative gains across investors will lead to an increase in the size of the financial industry and a reduction in real interest rates, but may not lead to increased physical investment. The reason is that if investment is specific to blueprints, increased displacement raises the risk that these investments may render themselves unprofitable. This may help explain why the low real rates observed in recent decades did not lead to a substantial increase in investment but rather an expansion of the financial industry.

2 Empirical motivation

Figure 1 illustrates the growth in alternative assets over the last decade. An obvious conclusion is that the alternative asset management industry in the form of private equity, venture

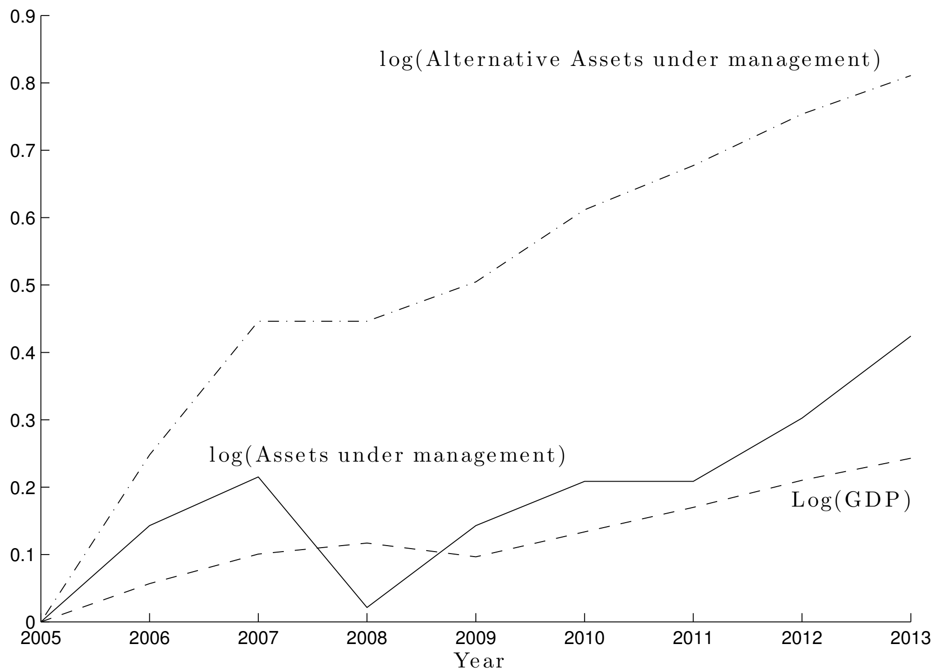

Figure 1: Cumulative first differences in the logarithm of GDP, assets managed by the financial industry, and alternative assets in the form of private equity, venture capital, and commercial real estate. Source: McKinsey (2015).

capital, and commercial real estate has grown much faster than either GDP or total assets under management in the economy. This substantial growth in the size of alternative investments coincided with substantial growth in other forms of alternative investments (such as commodities or hedge funds) and with a protracted period of declining real interest rates, trends that have been documented repeatedly in the literature.

Our goal in this paper is to explain these trends not as isolated phenomena, but rather as emanating from a common source, namely the increased incidence of "disruptive growth", or more accurately, redistributive growth.

To motivate these notions we we point to Figure 2. This figure shows that even though aggregate dividends and aggregate consumption share a common trend, the dividends-per-share of the S&P 500 follow a markedly slower growth path. (The same conclusion holds if we use the dividends accruing to the CRSP-value weighted portfolio). The difference in growth rates between aggregate dividends and dividends per share is approximately 2% per year. This discrepancy can be largely attributed to a dilution effect arising every time new companies enter the index. Indeed, in order to remain representative of the stock market,

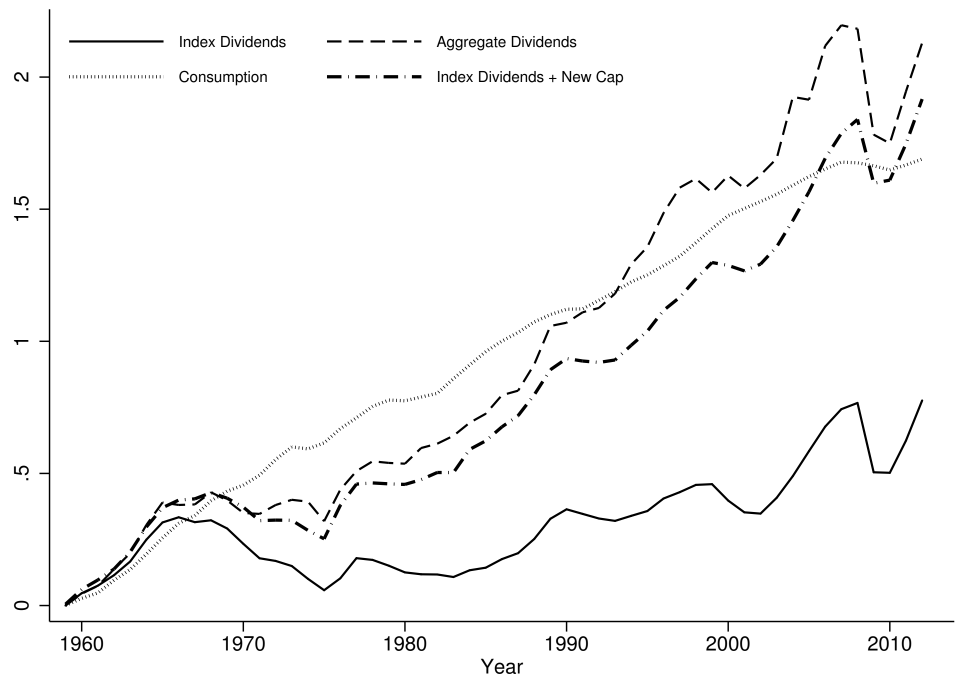

Figure 2: Total real logarithm of S&P 500 dividends per share, real log-aggregate consumption and real log-aggregate dividends. The CPI is used as a deflator for all series. The line "Index Dividends + New Cap" is equal to real log-dividends per share plus the cumulative (log) gross growth in the shares of the index that are due to the addition of new firms. Sources: R. Shiller's website, FRED, Personal Dividend Income series, and CRSPSift.

an index is re-constituted every year to include new additions. This process has an effect similar to dilutive equity issuance. Just like a cooperation would have to issue new shares to purchase a new company, thus diluting the existing shareholders, the same happens at the level of an index: Every time the market capitalization of the index increases due to the addition of new firms, index maintenance requires that the so-called "divisor" of the index (also called the "shares" of the index) be adjusted upward to keep the ratio of the market capitalization of the index to the number of shares in the index unchanged. The point of this adjustment is to ensure that the strategy of simply holding the market portfolio is self-financing, and the "dividends-per-share" of the index correspond to the cash flows of a self-financing strategy.

To verify that new company additions are the primary reason for the different trend followed by dividends-per-share, Figure 2 shows that if we add back to the time series of dividends-per-share all the share adjustments that have occurred due to new company

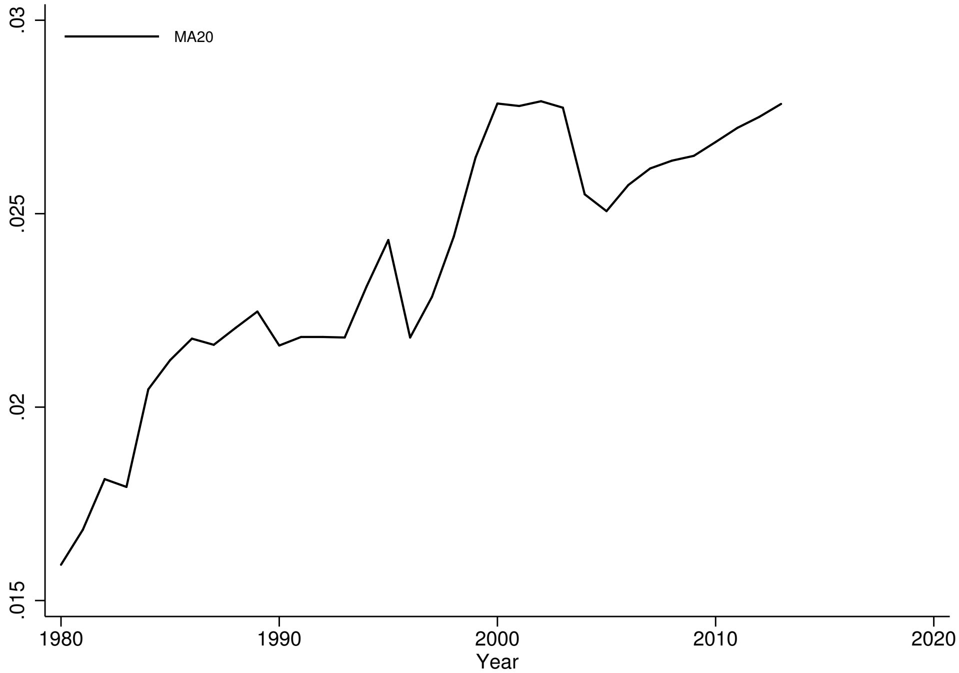

Figure 3: 20-year moving average (log) gross growth in the shares of the index that are due to the addition of new firms.

additions, then the resulting series comes very close to the NIPA aggregate dividend series.

A point not immediately visible from Figure 2 is that the share adjustment rate due to new company additions, which equals the ratio of the dollar value of index additions to the total market capitalization of the index, has been steadily increasing over time. Figure 3 shows this by plotting the 20-year moving average of that ratio. An immediate corollary is that since the growth rates of aggregate consumption and dividends remain roughly constant throughout this period, the increased incidence of share additions implies a widening gap between the dividends accruing to pre-existing and new firms.

The distinction between aggregate dividends and dividends-per-share is significant for asset pricing models, since only the latter series corresponds to the cash-flows of a self-financing strategy. Hence the usual asset pricing formulas (expressing the price of the stock market as the sum of discounted cash flows) would apply only if one uses the latter series of cash flows but not the former.

Besides drawing attention to modeling the cash flows of the market portfolio as being a declining fraction of aggregate consumption, the above observation is also significant for another reason. If the addition of a new firm to the market portfolio results in transfers

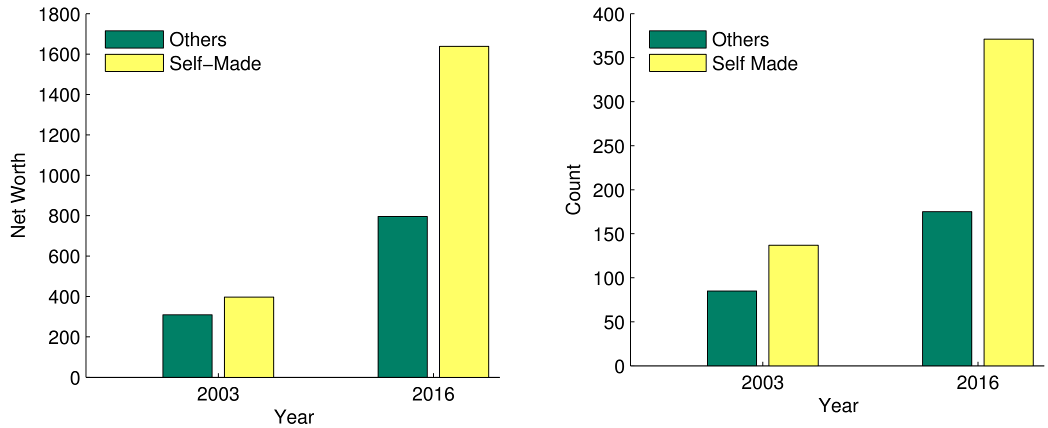

Figure 4: Wealth of US billionaires who inherited their wealth and those who were self-made (left figure). Number of US billionaires who inherited their wealth and number of billionaires who are self made. Source: Forbes 2003, 2016.

from the marginal agent to other agents (either new cohorts of "self made" entrepreneurs or existing investors who happened to invest early on in these newly created firms), then this will impact the stochastic discount factor of the marginal agent.

The data offer some evidence in that direction. Figure 4 compares the total wealth in the hands of billionaires that inherited their wealth with the wealth of "self-made" entrepreneurs. In 2003 (the earliest electronically available cross section in Forbes) the estimated net worth of self made billionaires is roughly comparable to that of billionaires who inherited their wealth. By 2016, the net worth in the hands of self-made billionaires is roughly twice as large as the wealth of billionaires who inherited their wealth.

In addition, Table 1 shows that a billionaire who inherited her wealth has about the same wealth as a self-made billionaire in 2016. Moreover, for ages between 45 and 65 a self-made billionaire tends to have a higher net worth than someone who inherited their wealth. This wasn't always the case. In 2003, a self-made billionaire had on average about 31% less wealth than a billionaire who inherited her wealth, with a t-stat of about -1.9. The difference is even larger for older billionaires.

Figure 5 shows that it is not only the means that are the same, the entire distribution of billionaire wealth looks identical.

To ensure that these conclusions are not driven by measurement error in the characterization of a billionaire as self-made or in the determination of their net worth, we confirm that we obtain similar results whether we use the Forbes database or the recently avail-

Figure 5: Kernel-smoothed density of log net worth for self made billionaires and billionaires who inherited their wealth. Sources: Forbes 2016 and Wealth-X 2016

able Wealth-X database. Wealth-X is a professionally maintained database of ultra-high-net-worth (UHNW) individuals. The company employs 170 employees, who collect data pertaining to an individual's publicly disclosed transactions, holdings, philanthropy, large purchases, board memberships, professional and family ties, etc., and aggregate them into a detailed "folder" for that individual.

Taken together, Table 1 and Figure 5 imply that self-made billionaires must have experienced a wealth growth that was higher than the wealth growth of pre-existing billionaires. To make this statement mathematically precise, express a billionaires wealth growth as $\log(W_T) = \log(W_0) + u$ where u combines the cumulative rate of growth in the value of a billionaire's assets net of her consumption-to-wealth ratio. Since the distribution of $\log(W_T)$ (conditional on W_T being above a billion) appears independent of $\log(W_0)$, this means that the distribution of u for self-made and non self-made billionaires cannot be the same. Indeed, the distribution of u for self-made billionaires must contain a positive location shift compared to the one for billionaires who inherited their wealth.

This could be driven by one of two forces: Either self-made billionaires "draw" their asset growth rate from a distribution that is shifted to the right (compared to that of billionaire

	Source: Wealth-X 2016		
	(1) All Ages	(2) 45 < Age < 65	(3) Age > 65
Self-Made Dummy	-0.029	0.049	-0.150
	(0.087)	(0.109)	(0.155)
Number of Observations	413	165	215
R^2	0.000	0.001	0.006
	Source:Forbes 2016		
	(1) All Ages	(2) 45 < Age < 65	(3) Age > 65
Self-Made Dummy	-0.010	0.124	-0.148
	(0.095)	(0.110)	(0.177)
Number of Observations	465	197	233
R^2	0.000	0.005	0.004
	Source: Forbes 2003		
	(1) All Ages	(2) 45 < Age < 65	(3) Age > 65
Self-Made Dummy	-0.310	-0.324	-0.492
	(0.165)	(0.240)	(0.264)
Number of Observations	176	83	73
R^2	0.027	0.028	0.062

Table 1: Regressions of billionaire's log(net worth) on a dummy variable taking the value one if the billionaire is characterized as self made in the respective data set. Standard errors in parentheses

	(1) lnphil	(2) lnphil	(3) lnphil	(4) lnphil	(5) lnphil	(6) lnphil
lnw	1.157***	1.149***	1.199***	1.191***	0.983***	0.982***
	(0.151)	(0.149)	(0.341)	(0.334)	(0.166)	(0.164)
SelfMade		0.234		0.0953		0.278
		(0.269)		(0.517)		(0.313)
Constant	-8.312*	-8.292*	-9.691	-9.582	-4.160	-4.320
	(3.338)	(3.313)	(7.424)	(7.324)	(3.679)	(3.640)
Observations	258	258	86	86	154	154
R^2	0.201	0.204	0.184	0.184	0.174	0.178

Table 2: Regressions of log(Philanthropy) on log(net worth) and a dummy variable taking the value one if the billionaire is listed as self made. Regressions (1) and (2) pertain to the entire sample, (3) and (4) to billionaires between 45 and 65 and (5) and (6) to billionaires above 65.

heirs), or they have a lower expenditure-to-wealth ratio. Table 2 suggests that the second explanation is unlikely. Using data from the Wealth-X database, we regress an individual's publicly known log philanthropic expenditure over her life-time on her estimated log-net worth and a dummy variable taking the value one if the billionaire is listed as "self-made". This table shows two things: First the coefficient on log net worth is essentially equal to one. Since it is reasonable to expect that philanthropy should be proportional to an individual's net worth, this is re-assuring: It suggests that net worth is measured reasonably well, and does not just capture "paper money;" instead we see it reflected in an easily observed expenditure component. The second conclusion from the table is that, if anything, self-made individuals spend a slightly higher fraction of their wealth on philanthropy compared to billionaires who have inherited their wealth. Even though this observation pertains only to philanthropy, it is suggestive that differences in expenditure patterns do not seem a likely candidate for the observed differences in wealth growth rates.

We summarize the pieces of the empirical evidence that motivates our model as follows: 1) Recent decades have seen an increase in portfolio allocations to alternative asset classes, real assets, and a simultaneous drop in the real interest rate. 2) The addition of new firms to the market portfolio acts in a manner similar to dilution for existing investors. 3) The rate of these additions has progressively increased in recent decades. 4) At the same time, the relative wealth of self-made billionaires as a fraction of billionaire wealth appears to have increased. 5) The distribution of wealth growth of self-made billionaires appears to be first-order-stochastically dominating that of billionaires who inherited their wealth.

This evidence suggests that the gains of new firm creation are not equally shared in the population. The assumption of representative-agent models, whereby the gains from new firm creation are equally shared by the representative agent (or, more generally, perfectly shared by all market participants), is a far cry from reality; in such a world, all wealth growth rates should be equal, and firm creation should not make anyone wealthy.

Motivated by the above observations, in the next section we build a dynamic general equilibrium model that explicitly allows for heterogenous wealth growth rates amongst market participants.

3 Model

3.1 Agent's preferences and demographics

We consider a model with discrete and infinite time: $t = \{..., 0, 1, 2, ...\}$. The size of the population is normalized to one. At each date a mass λ of agents are born, and a mass λ dies so that the population remains constant. We denote by $V_{t,s}$ the utility at time t of an agent born at time s. Preferences are logarithmic:

$$\log V_{t,s} = \log c_{t,s} + \beta (1 - \lambda) \mathrm{E}_t \log V_{t+1,s}, \tag{1}$$

where $\beta \in (0, 1)$ is the agent's subjective discount factor, and $c_{t,s}$ is the agent's consumption at time t. These preferences imply that the representative agent has an intertemporal elasticity of substitution (IES) equal to one and a risk aversion equal to one. These preferences are convenient for obtaining closed form solutions. We consider extensions to allow for general risk aversion and IES later.

3.2 Technology

Output is produced by a representative (competitive) final-good firm, which uses two categories of inputs: (a) labor and (b) a continuum of intermediate goods. Letting L_t^F denote the efficiency units of labor that enter into the production of the final good, A_t the number of intermediate goods available at time t, and $x_{j,t}$ the quantity of intermediate good j used in the production of the final good, the production function of the final-good producing firm is

$$Y_t = \left(L_t^F\right)^{1-\alpha}\left[\int_0^{A_t} x_{j,t}^{\alpha}dj\right] \tag{2}$$

At each point in time the representative final-good firm chooses L_t^F and $x_{j,t}$ to maximize its profits

$$\pi_t^F = \max_{L_t^F, x_{j,t}}\left\{Y_t - \int_0^{A_t} p_{j,t}x_{j,t}dj - w_t L_t^F\right\}, \tag{3}$$

where $p_{j,t}$ is the price of intermediate good j at time t and w_t is the prevailing wage at time t.

The intermediate goods $x_{j,t}$ are produced by monopolistically competitive firms that own nonperishable blueprints to the production of these goods. The production of each intermediate good requires one unit of labor per unit of intermediate good produced, so that $L_t^I = \int_0^{A_t} x_{j,t}dj$,where L_t^I is the total amount of labor used in the intermediate goods sector.

The price $p_{j,t}$ is set by the intermediate-good firm j to maximize their profits

$$\pi_t^I = \max_{x_{j,t}}\left\{\left(p_{j,t} - w_t\right)x_{j,t}\right\}.$$

Labor is supplied inelastically by workers (to be introduced shortly) and is in fixed total supply equal to one.

We state some standard results associated with this production setup, which are useful for our purposes. We refer to Gârleanu et al. (2012) for proofs.

Lemma 1 *The share of labor directed to intermediate goods L_t^I is constant, and so is L_t^F. The optimal amount of intermediate good j produced is $x_{j,t} = \frac{L_t^I}{A_t}$, and output is proportional to $A_t^{1-\alpha}$:*

$$Y_t \propto A_t^{1-\alpha}. \tag{4}$$

The profits of final-good firms are zero, and the total profits of intermediate-goods firms equal

$$A_t\pi_t = \alpha\left(1-\alpha\right)Y_t, \tag{5}$$

where we have dropped the superscript I to write π_t instead of π_t^I. Accordingly, the profits accruing to each blueprint are equal to

$$\pi_t = \frac{\alpha\left(1-\alpha\right)Y_t}{A_t} \propto A_t^{-\alpha}. \tag{6}$$

This lemma captures some of the key implications of the by now standard model of expanding varieties. An increasing number of blueprints A_t raises total output Y_t (equation (4)), but the profits per blueprint decline (equation (6)). This is the sense in which this simple production specification captures the idea of displacement of old blueprints by new ones.

We would like to point out here that, even though we opted for a basic Romer-style production specification, the specific production assumptions (whether they are of the Romer type or the quality-ladder type) are irrelevant for the intuitions we develop in this paper.

3.3 New agents: Workers, entrepreneurs, and new products

The measure λ of newly born agents are of two types. A fraction θ are entrepreneurs and a fraction $1-\theta$ are workers. We assume that workers supply one unit of labor inelastically throughout their life. Since workers are not the focus of the paper, we assume that they are "hand-to-mouth" consumers, i.e., their wage income equals their consumption period-by-period. This assumption is not essential for the results, and we relax it in a later section.

The focus of the paper is on business owners (entrepreneurs). Entrepreneurs are born with the right to new blueprints for new intermediate goods. At the time of their birth, entrepreneurs are uncertain about the number of blueprints that they will be receiving.

To model this uncertainty we assume that, once born, a mass $\lambda\theta di$ of entrepreneurs is assigned to every "location" $i \in [0,1)$ on a circle, so that $\int_0^1 \lambda\theta di = \lambda\theta$ is the total mass of newly born entrepreneurs each period.

Each period a total mass

$$\Delta A_{t+1} = A_{t+1} - A_t = \eta A_t \Gamma_{t+1} \tag{7}$$

of new blueprints arrives, where Γ_{t+1} is a gamma distributed variable with shape parameter a and rate parameter b and η is a constant. Since the gamma distribution is infinitely divisible, we will write $\Gamma_{t+1} \equiv \int_{i\in[0,1]} d\Gamma_{i,t+1}$, where $d\Gamma_{i,t+1}$ denotes the increments of a gamma process and captures the increment in the mass of blueprints that will be assigned to location i at time $t+1$.[1]

Since the Gamma process is not commonly used in economics, we summarize briefly some of its properties. To build intuition, it is most useful to split the interval $[0,1]$ into N equal intervals, and think of the Gamma process at the location $\frac{k}{N}$ as a sum of gamma-distributed

[1]For technical reasons, we think of new entrepreneurs as indexed by $(i,j) \in [0,1) \times [1,1]$, with, for all j, (i,j) assigned to location i and receiving the same number of blueprints $\eta A_t d\Gamma_{i,t+1}$.

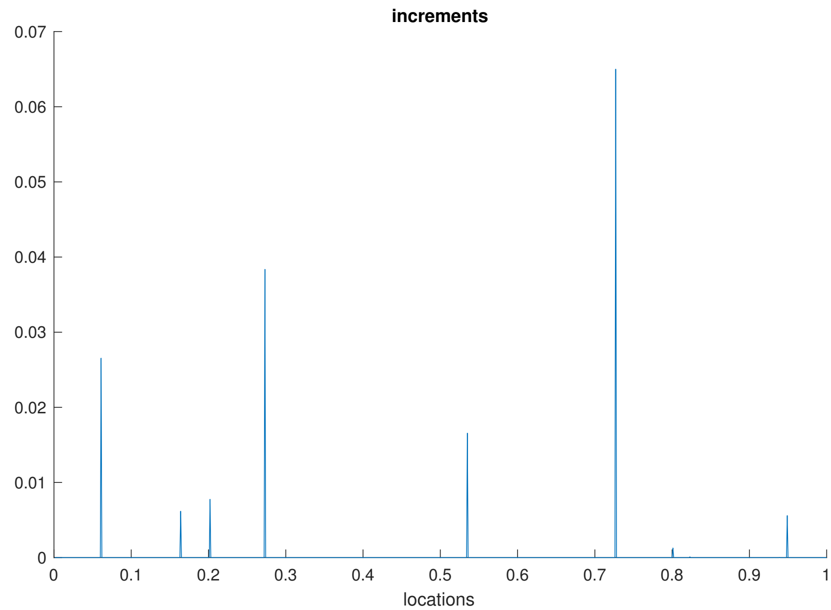

Figure 6: An illustration of the increments ξ_i, for the case $N = 1000$, $a = 1$, $b = 2$.

increments $\xi_{\frac{n}{N}}$,

$$\sum_{n=1}^{k} \xi_{\frac{n}{N}}, \tag{8}$$

where the pdf of the increment ξ_i is given by

$$Pr(\xi_i \in dx) = \frac{b^{\frac{a}{N}}}{\Gamma\left(\frac{a}{N}\right)} x^{\frac{a}{N}-1} e^{-bx}\, dx. \tag{9}$$

The parameters $\frac{a}{N}$ and b are sometimes refered to as the "shape" and the "rate" of the gamma distribution, and $\Gamma\left(\frac{a}{N}\right)$ is the gamma function evaluated at $\frac{a}{N}$. The increments ξ_i are indepenent of each other, and the properties of the gamma distribution imply that

$$Pr\left(\sum_{n=1}^{N} \xi_{\frac{n}{N}} \in dx\right) = \frac{b^a}{\Gamma\left(a\right)} x^{a-1} e^{-bx}\, dx, \tag{10}$$

which is the distribution of a gamma variable with shape a and rate b.

Using the gamma process is technically attractive for our purposes, since it captures in a stylized way the fact that entrepreneurship is very risky. This is illustrated in Figure 6. The figure shows a sample of increments ξ_i for the case $N = 1000$. The figure illustrates that these increments tend to be close to zero for most of the locations; however, a small subset of random locations exhibit big spikes of random height. From an economic point of view, this means that only the lucky few entrepreneurs who happen to find themselves in the locations exhibiting the large spikes obtain a valuable allocation of blueprints.

The limit of the variables given by (8) as the number of locations N goes to infinity is a Gamma process. It is a positive and increasing process, whose paths are not continuous (they are only left continuous with right limits).[2] This implies that any given location is unlikely to receive a non-trivial allocation of blueprints. However because the process $\Gamma_{i,t+1}$ is a discontinuous function of i, a zero-measure of locations may receive a strictly positive measure of blueprints, and the entrepreneurs who find themselves in these locations become spectacularly wealthy.

Before proceeding, we would like to note that this extreme inequality setup is mostly for illustrative purposes and technical convenience. Less extreme distributions[3] would not affect the economic insights, as long as we preserve some notion of distributional risk.

A final crucial assumption is that no agent knows the realization of the gamma process path at time t. Everyone is trading behind the "veil of ignorance" about which locations on the circle will obtain the valuable blueprints and which ones will obtain the useless ones. Newly arriving entrepreneurs are therefore eager to share that risk by selling shares to investors on the market before this uncertainty is resolved. These shares entitle investors to a fraction υ of the profits that will be produced by the newly arriving firms in perpetuity. A fraction $1-\upsilon$ is "inalienable," a reduced form way of capturing incentive effects of equity retention.

3.4 Markets

At each point in time, an investor can trade a zero net-supply bond. We follow Blanchard (1985) and assume that agents can also trade annuities with competitive insurance companies that break even. These annuity contracts entitle an insurance company to collect the wealth $W_t^{(j)}$ of an agent j in the event that she dies at time t and in exchange provide her with an income stream $\lambda W_t^{(j)}$ while she is alive. We refer to Blanchard (1985) for further details.

Investors at time t can trade costlessly in the shares of all companies that were created prior to time t. Per blueprint, all such companies make the same profits π. For future use,

[2] The Gamma process is, however, continuous in probability. This means that, for any $\varepsilon > 0$, $\lim_{\delta\to 0} \Pr(|\Gamma_{i+\delta,t+1} - \Gamma_{i,t+1}| > \varepsilon) = 0$.

[3] Something as simple as assuming that locations are finite rather than a continuum would produce less extreme distributions without affecting the economic intuitions.

we denote by Π_t the value of the future stream of profits from the representative blueprint:

$$\Pi_t = \mathrm{E}_t \left[\sum_{t+1}^{\infty} \frac{M_s^i}{M_t^i} \pi_s \right], \tag{11}$$

with M_s^i the marginal-utility process of a given investor.

The shares of new firms are introduced to the market via intermediaries as follows: Every investor at time t is assigned to a location i on the circle $[0, 1)$. At each location i there is a representative competitive intermediary, who purchases an equally weighted portfolio of shares of companies located in an arc of length Δ_i centered at location i. The intermediary then offers the portfolio for purchase to the investors in location i. Intermediation requires resources equal to $\psi A_t^{1-\alpha} f(\Delta_i)$ per share. (We make the cost proportional to $A_t^{1-\alpha}$ to ensure that the cost of intermediation is a stationary fraction of the size of the aggregate economy). Hence, to break even, the intermediary needs to sell each share of the portfolio at a price $\frac{1}{\Delta_i} \int_{i-\frac{\Delta_i}{2}}^{i+\frac{\Delta_i}{2}} P_t^{(j)} dj + \psi A_t^{1-\alpha} f(\Delta_i)$, where $P_t^{(j)}$ is the price of a newly created firm in location j. Assuming that there exist location-invariant equilibria such that $P_t^{(j)} = P_t$, the price of a portfolio share is simply $P_t + \psi A_t^{1-\alpha} f(\Delta_i)$. To simplify notation, from now on, we will guess that there exist equilibria with $P_t^{(j)} = P_t$, and will then verify their existence in the next section.

Figures 7 and 8 illustrate how intermediaries can facilitate risk sharing in this economy. By purchasing an equal-weighted portfolio of shares on an arc of length Δ, the intermediaries are able to "smooth out" the spikes of the gamma process. Indeed, as the Figure illustrates, they can offer their investors a portfolio of blueprints that has the same mean as the number of blueprints that arrive in each location, but is second-order stochastically dominant. Specifically, by using properties of the gamma distribution, one can show that $\frac{1}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_j$ is gamma distributed with shape $a\Delta$ and rate $b\Delta$, and accordingly it has mean equal to $\frac{a}{b}$ and standard deviation equal to $\frac{\sqrt{a}}{b\sqrt{\Delta}}$. Further, it holds that, if $\Delta_2 > \Delta_1$, then $\frac{1}{\Delta_2} \int_{i-\frac{\Delta_2}{2}}^{i+\frac{\Delta_2}{2}} d\Gamma_j \succcurlyeq^2 \frac{1}{\Delta_1} \int_{i-\frac{\Delta_1}{2}}^{i+\frac{\Delta_1}{2}} d\Gamma_j$, where $\succcurlyeq^2$ denotes second-order stochastic dominance.

The arc-length Δ has an intuitive interpretation as a correlation coefficient. Specifically, the correlation between the blueprints accruing to a given fund $\frac{1}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_j$ and total displacement $\int_{i-\frac{1}{2}}^{i+\frac{1}{2}} d\Gamma_j$ is $\sqrt{\Delta}$.

Intermediaries in each location are competitive and in an effort to attract investors they determine Δ in a way that maximizes investor welfare. Moreover, the assumption of perfect competition ensures that intermediaries make no profits. Accordingly, they act as simple pass throughs, enabling existing investors access to the newly created firms albeit at a cost.

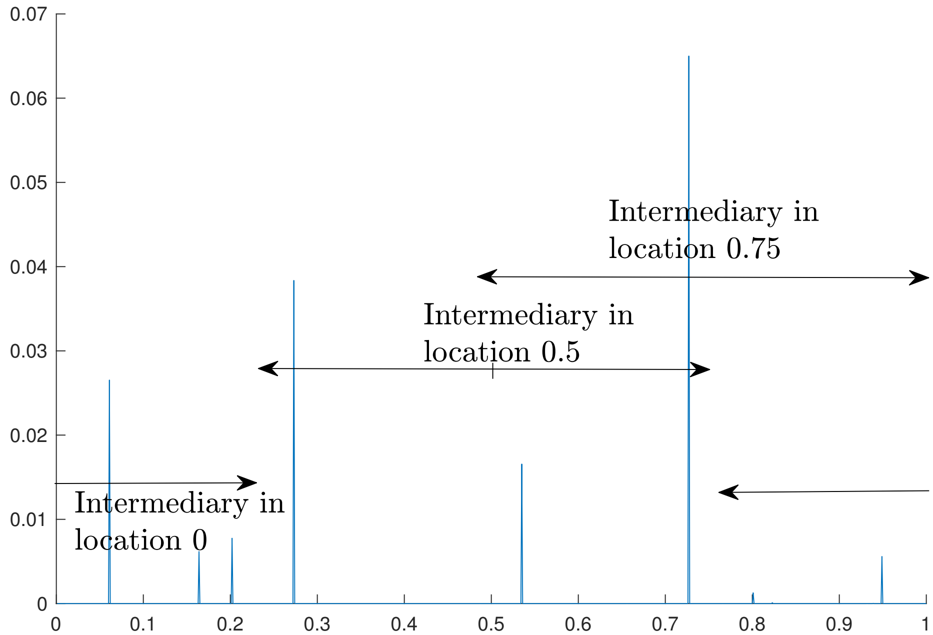

Figure 7: An illustration of how intermediaries help with risk sharing. The increments are the same as in Figure 6, and $\Delta = 0.5$. The intermediary in position 0.5 provides an equal weighted portfolio of the increments in $[0.25, 075]$. The intermediary in position 0.75 averages the increments in $[0.5,\ 1]$, while the intermediary in position 0 averages the increments in $[0, 0.25] \cup [0.75, 1]$.

To allow for heterogeneity in the returns of existing investors, we assume that $f(1) = \infty$, implying that Δ lies in the interior of $[0, 1]$.

We conclude this section with two comments. First, the notion of a "location" should not be understood geographically. It is simply a convenient device to produce heterogenous returns across investors. Second, since the gamma process has independent increments, it is immaterial whether investors invest in a single arc of length Δ or a set of non-contiguous arcs of total length Δ. Our construction only requires that these sets satisfy rotational symmetry.

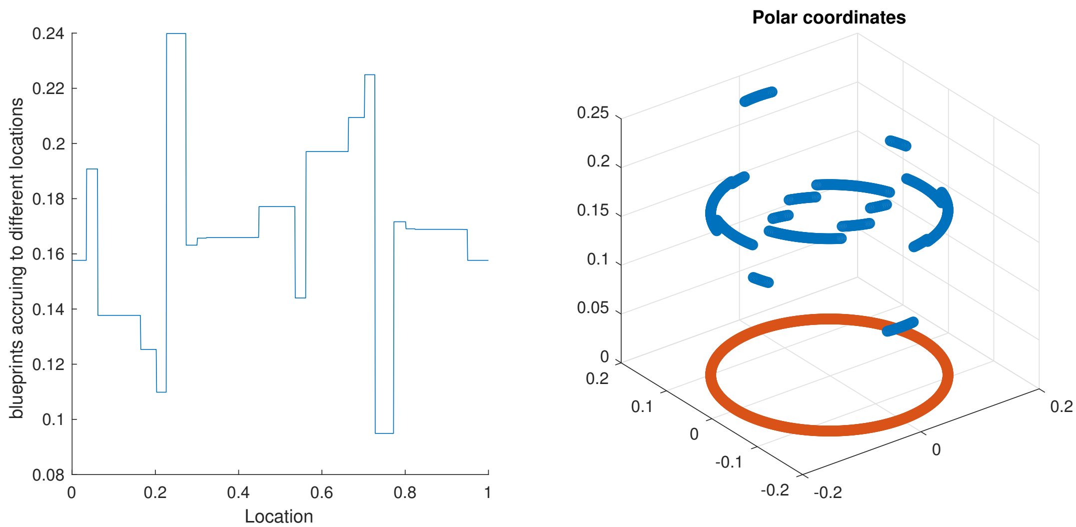

Figure 8: The distribution of equal weighted returns. The left figure depicts the blueprints accruing to the portfolio formed by the intermediary in each location i, which is simply an equal weighted average of the blueprints accruing to locations in an arc Δ around the intermediary's location. The right figure is identical to the left figure except that the results are now depicted in polar coordinates.

3.5 Budget constraints

With these assumptions, the dynamic budget constraint of an investor who resides in location i, can be expressed as

$$W_t^i = S_t^{E,i} A_t \Pi_t + B_t^i + S_t^{N,i}\left(P_t + \psi A_t^{1-\alpha} f\left(\Delta\right)\right) + c_t^i, \tag{12}$$

$$W_{t+1}^i = S_t^{E,i} A_t \left(\Pi_{t+1} + \pi_{t+1}\right) + (1 + r_t^f) B_t^i + S_t^{N,i} \frac{\eta \upsilon}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} \left(\Pi_{t+1} + \pi_{t+1}\right) A_t \, d\Gamma_{i,t+1} + \lambda W_{t+1}^i,$$

where $S_t^{E,i}$ are the shares of the (representative) firm that is already traded at time t, B_t^i is the amount invested in, r_t^f the interest rate, and $S_t^{N,i}$ is the number of shares purchased in the intermediary-provided portfolio of newly created firms. We normalize the supply of shares of all firms to unity. A convenient way to express (12) is

$$\frac{W_{t+1}^i}{W_t^i} = \left(\frac{1 - \frac{c_t^i}{W_t^i}}{1 - \lambda}\right) \left(\phi_B^i \left(1 + r_t^f\right) + \phi_2^i R_{t+1}^E + \phi_N^i R_{t+1}^{N,i}\right), \tag{13}$$

where $\phi_B^i \equiv \frac{B_t^i}{W_t^i - c_t^i}$, $\phi_E^i \equiv \frac{S_t^{E,i} A_t \Pi_t}{W_t^i - c_t^i}$, and $\phi_N^i = \frac{S_t^{N,i}\left(P_t + \phi A_t^{1-\alpha} f(\Delta)\right)}{W_t^i - c_t^i}$ are the post-consumption wealth shares invested by investor i in bonds, existing firms, and newly arriving firms respectively, and $R^E_{t+1} \equiv \frac{\Pi_{t+1}+\pi_{t+1}}{\Pi_t}$ and $R^{N,i}_{t+1} \equiv \frac{\frac{\eta v}{\Delta} A_t \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} (\Pi_{t+1}+\pi_{t+1}) d\Gamma_{i,t+1}}{P_t + \psi A_t^{1-\alpha} f(\Delta)}$ are the gross returns of existing firms and the portfolio of newly arriving firms that investor i invests in.

An important observation about (13) is that as long as $P_t^{(j)} = P_t$ for all $j \in [0,1)$, the choices ϕ^i_B, ϕ^i_E, and ϕ^i_N, the choice of Δ_i, and the choice of $\frac{c^i_t}{W^i_t}$ are the same for all investors, irrespective of their level of wealth and the location where they reside at time t, which simplifies the solution and analysis of the model.

To ensure that $P_t^{(j)} = P_t$ for all $j \in [0,1)$ we make one final assumption, namely that investors re-locate once returns in period $t+1$ have materialized, so that the total wealth of all the investors positioned in each location becomes equal across locations.[4]

The motivation behind this assumption is free entry into locations. Conditional on all funds offering the same arc-length $\Delta_i = \Delta$, and given the location-invariant nature of the distribution of new firms across the circle, it makes sense for investors to move to locations that offer lower prices. This outcome occurs when wealth moves across locations in such a way that the total wealth in every location is equalized. (In turn, this would make the choice of Δ_i location invariant, confirming the anticipation $\Delta_i = \Delta$.)

3.6 Location-invariant equilibrium

The definition of a location-invariant equilibrium is standard. Such an equilibrium is a collection of prices Π_t, p_t, and P_t, portfolio allocations ϕ_B, ϕ_E, and ϕ_N, a choice of Δ, and consumption processes for all agents c^j_t such that a) Given prices, ϕ_B, ϕ_E, ϕ_N, Δ, and c^j_t are choices that maximize (1) subject to (13), b) the consumption market clears: $\int_j dc^j_t = A_t\pi_t - \psi A_t^{1-\alpha} f(\Delta)$, c) the markets for all shares (both new and existing) clear: $\int_j dS^{E,j}_t = \int_j dS^{N,j}_t = 1$, and d) the bond market clears: $\int_j dB^j_t = 0$.

4 Solution

Next we construct an equilibrium that is both location-invariant, time-invariant and symmetric, in the sense that all agents choose the same portfolio. Specifically, we guess that there exist an equilibrium whereby $\phi_B = 0$, and the portfolio shares ϕ_E and ϕ_N, the interest rate r^f, the participation arc Δ, the valuation ratios $\pi^E \equiv \frac{\Pi_t}{\pi_t}$ and $P^N \equiv \frac{P_t}{A_t\pi_t}$, and the consumption-to-wealth ratio $c \equiv \frac{c^i_t}{W^i_t}$ are the same for all agents and constant across

[4] Mathematically, such a re-location is always possible; one of the infinitely many ways to achieve it is to assign the investor with wealth $W_t^{(j)}$ to location $F^{-1}(W_t^{(j)})$, where $F(\cdot)$ is the wealth distribution.

time. After computing explicit values for the constants that support such an equilibrium, we provide sufficient conditions for its existence.

It is best to construct the equilibrium by specializing to the case where investors have logarithmic preferences, which we assume in this section. Later we show how to extend the results to cases where investors have constant relative risk aversion different than one, or more general recursive preferences.

Maintaining the supposition that $\frac{\Pi_t}{\pi_t}$ is a constant equal to P^E, the return R^E_{t+1}, can be expressed as

$$\begin{aligned} R^E_{t+1} &= \frac{\pi_{t+1}+\Pi_{t+1}}{\Pi_t} = \frac{\pi_{t+1}}{\pi_t}\left(\frac{1+P^E}{P^E}\right) = \left(\frac{1+P^E}{P^E}\right)\left(\frac{A_{t+1}}{A_t}\right)^{-\alpha} \\ &= \left(\frac{1+\pi^E}{P^E}\right)\left(1+\eta\Gamma_{t+1}\right)^{-\alpha}. \end{aligned} \tag{14}$$

Moreover, using the supposition that $\frac{P_t}{A_t\pi_t}$ is a constant equal to P^N and letting $\delta \equiv \frac{\psi A_t^{1-\alpha}}{A_t \pi_t}$, the return on an equal-weighted portfolio of newly created firms over an arc Δ is

$$R^N_{t+1} = \frac{\upsilon A_t\left(\Pi_{t+1}+\pi_{t+1}\right)\frac{\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}}{A_t\pi_t\left(P^N+\delta f\left(\Delta\right)\right)} \tag{15}$$

$$= R^E_{t+1}\frac{P^E}{P^N+\delta f\left(\Delta\right)}\left(\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}\right). \tag{16}$$

The following proposition provides a set of necessary conditions associated with a symmetric, location-invariant equilibrium.

Proposition 1 *Assuming that a location-invariant, time-invariant, and symmetric equilibrium exists, the unique values of of ϕ_k and c that support such an equilibrium are $\phi_B = 0$,*

$$\phi_E = E\left[\left(1+Z\right)^{-1}\right] \tag{17}$$

$$\phi_N = 1-\phi_E, \tag{18}$$

with

$$Z \equiv \frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t+1} \tag{19}$$

gamma distributed with shape $a\Delta$ and rate $\frac{b\Delta}{\eta\upsilon}$, and

$$c = 1-\beta\left(1-\lambda\right). \tag{20}$$

The equilibrium values of P^E and P^N are

$$P^E = \phi_E \frac{1-\delta f(\Delta)}{1-\beta(1-\lambda)} \beta(1-\lambda), \tag{21}$$

$$P^N = (1-\phi_E) \frac{1-\delta f(\Delta)}{1-\beta(1-\lambda)} \beta(1-\lambda) - \delta f(\Delta), \tag{22}$$

and the interest rate equals

$$1 + r^f = \frac{E\left[(1+Z)^{-1}\right]}{E\left[\left(R^E_{t+1}\right)^{-1}(1+Z)^{-1}\right]}. \tag{23}$$

Finally, the equilibrium value of Δ is given by the solution of the equation

$$[1-\beta(1-\lambda)] \frac{\delta f'(\Delta)}{1-\delta f(\Delta)} = \beta(1-\lambda) \frac{\partial E[\log(1+Z)]}{\partial \Delta}. \tag{24}$$

Proposition 1 contains an explicit description of a symmetric, time- and location-invariant equilibrium.

We analyze the properties of the equilibrium in steps. First, we derive the implications of the equilibrium for risk sharing both within and across cohorts of entrepreneurs. Then we discuss implications for the size of the financial industry and its relation to the equilibrium expected excess returns of existing firms and new ventures.

4.1 Risk sharing implications

For presentation purposes, it is convenient to start the analysis by treating Δ not as a choice variable, but rather as an exogenous parameter.

To derive the implications of the model for risk sharing between and accross investor cohorts, we start with the following lemma.

Lemma 2 *Aggregate wealth growth is given by*

$$\frac{W_{t+1}}{W_t} = (1+\eta\Gamma_{t+1})^{1-\alpha}, \tag{25}$$

while an individual investor's wealth growth (conditional on survival) is given by

$$\frac{W^i_{t+1}}{W^i_t} = \frac{W_{t+1}}{W_t}\left(\frac{1}{1-\lambda}\right)\left(\frac{1+P^E}{1+P^E+P^N}\right)\left(\frac{1+\eta\upsilon\Gamma_{t+1}}{1+\eta\Gamma_{t+1}}\right) X_{i,t+1}, \tag{26}$$

where

$$X_{i,t+1} \equiv \frac{1+\eta\upsilon\Gamma_{t+1}\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}}{1+\eta\upsilon\Gamma_{t+1}} \tag{27}$$

$$dL_{j,t+1} \equiv \frac{d\Gamma_{j,t+1}}{\Gamma_{t+1}}. \tag{28}$$

Equation (26) reveals that risk is imperfectly shared both within and across cohorts. The lack of within-cohort risk sharing is captured by the term $X_{i,t+1}$, which reflects heterogenous investment returns experienced by existing agents. This is driven by their inability to invest in all available ventures. Indeed, for $\Delta = 1$ the term $X_{i,t+1}$ becomes one, and the within-cohort lack of risk sharing disappears.

However, this is not the only dimension along which risk is imperfectly shared. Even if $\Delta = 1$, equation (26) shows that individual wealth $\frac{W^i_{t+1}}{W^i_t}$ and aggregate wealth $\frac{W_{t+1}}{W_t}$ are not perfectly correlated as long as υ is less than one. The random term $\frac{1+\eta\upsilon\Gamma_{t+1}}{1+\eta\Gamma_{t+1}}$ captures the inter-cohort lack of risk sharing. It is driven by the fact that newly arriving entrepreneurs retain a fraction $1-\upsilon$ of new company shares. Indeed, for $\upsilon = 1$ the term $\frac{1+\eta\upsilon\Gamma_{t+1}}{1+\eta\Gamma_{t+1}}$ would disappear.

In summary, Δ controls the extent of intra-, while υ controls the extent of inter-cohort risk sharing. If $\Delta = \upsilon = 1$, then risk is perfectly shared both within and accross cohorts; individual wealth growth and aggegate wealth growth are perfectly correlated. However, even in that case individual and aggregate wealth growth differ by a negative constant. Indeed, aggregating the wealth growth of all investors surviving into $t+1$, we obtain

$$\log\left((1-\lambda)\frac{\int_i W^i_{t+1}di}{W_t}\right) - \log\left(\frac{W_{t+1}}{W_t}\right) = \log\left(\frac{1+\pi^E}{1+P^E+P^N}\right) < 0. \tag{29}$$

The negative constant reflects that the wealth owned by existing investors does not include new-firm endowments, which are nevertheless part of aggregate wealth.

4.2 Implications for the SDF

Since the wealth-to-consumption ratio is constant, our conclusions on wealth changes apply without modification to consumption changes of individual investors: an individual investor's consumption change is given by the right hand side of (26). With logarithmic utilities, the SDF M_t of an individual investor is given by

$$\begin{aligned}\frac{M^i_{t+1}}{M^i_t} &= \beta(1-\lambda)\left(\frac{W^i_{t+1}}{W^i_t}\right)^{-1} \\ &\propto (1+\eta\Gamma_{t+1})^{\alpha}\left(1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}\right)^{-1}.\end{aligned} \tag{30}$$

For the markets where all investors are participating (in particular, the market for existing stocks and the risk-free asset), any $\frac{M^i_{t+1}}{M^i_t}$ is a valid SDF, and so is $\frac{M_{t+1}}{M_t} \equiv \mathrm{E}\left[\frac{M^i_{t+1}}{M^i_t}|\Gamma_{t+1}\right]$.

By the properties of gamma distributed variables, the quantity $\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}$ is beta distributed and independent of Γ_{t+1}. One can provide an explicit closed form solution for $\frac{M_{t+1}}{M_t}$ as a function of Γ_{t+1} in terms of a hypergeometric function. For our purposes, we wish to investigate whether $\frac{M_{t+1}}{M_t}$ is increasing or decreasing in Γ_{t+1}:

Lemma 3 *$\frac{M_{t+1}}{M_t}$ is decreasing in Γ_{t+1} when risk sharing both across and within cohorts is perfect, that is, when $\Delta = 1$ and $\upsilon = 1$. However, $\frac{M_{t+1}}{M_t}$ is increasing in Γ_{t+1} when either υ or Δ is sufficiently small.*

Lemma 3 shows how risk sharing imperfections can determine whether the marginal utility of consumption of the representative investor rises or declines as the extent of displacement Γ_{t+1} increases. If risk is shared perfectly both within and across cohorts, then large realizations of Γ_{t+1} are "good news" for the representative investor. The gains in the value of the portfolio of new firms are enough to undo the losses on the existing assets owned by the investor. However, away from the perfect risk sharing limit, large realizations of Γ_{t+1} are "bad news." For instance, if risk is shared perfectly within cohorts but imperfectly across cohorts, then a fraction of the value of new ventures cannot be separated from the newly arriving cohort of agents. Hence, the losses on the portfolio of existing assets cannot be offset by the gains on the portfolio of new ventures. This is the key insight of Gârleanu et al. (2012). But even if risk is perfectly shared across cohorts, large realizations of Γ_{t+1} may be (unconditionally) perceived as states of high marginal utility ("bad states"). Conditional on Γ_{t+1} being high, the only certain outcome is a decline in the value of existing assets. Even though existing investors as a group obtain the gains of that growth, they do not know ex ante whether they will receive a large or a small allotment of the new firms. Since investors are risk averse, they assign greater weight to the event that they end up with a disproportionately small share of the gains from growth, and therefore they perceive a high realization of Γ_{t+1} as bad news. This intuition is reminiscent of the intuition put forth by Constantinides and Duffie (1996), Kogan and Papanikolaou (2014), and Garleanu et al. (2015).

Hence our model nests models of perfect risk sharing as well as of imperfect risk sharing, across and within cohorts, as special cases. However, the most important difference is that it proposes a view of the financial industry as a (costly) device to improve risk sharing, yielding joint predictions on how expected returns, the size of the financial industry, the interest rate, etc. change as, say, displacement increases.

4.3 Equilibrium excess returns

Having derived the equilibrium SDF we can now discuss the implications of the model for expected excess returns. We start with a comparison of the expected returns on new ventures as opposed to existing firms. We then discuss implications for the excess return on the stock market. Using (14), (16), and the results of Proposition 1 leads to

$$\mathrm{E}\left[\frac{R^N}{R^E}\right] = \frac{1-\phi_N(\Delta)}{\phi_N(\Delta)} \times \mathrm{E}\left[\frac{\eta \upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t+1}\right]. \tag{31}$$

The right-hand side of (31) contains two terms. The first term, $\frac{1-\phi_N(\Delta)}{\phi_N(\Delta)}$, is a declining function of ϕ_N, the share of aggregate wealth invested in new ventures. The second term has expectation equal to $\eta\upsilon\frac{a}{b}$, which is independent of Δ, and indeed of any endogenous model variable. Equation (31) shows that the expected return on new investments as compared to existing investments is negatively related to ϕ_N.

Lemma 4 *For any $\Delta \in [0,1]$ and any $\upsilon \in [0,1]$ it holds that $E\left[\frac{R^N}{R^E}\right] \geq 1$.*

Lemma 4 may appear counterintuitive at first pass. When ϕ_3 is small, then the bulk of investors' wealth is invested in existing assets, whose value is declining in aggregate displacement Γ_{t+1}. Accordingly, one would expect new assets to act as a partial hedge against capital losses on existing assets, and hence to have a low expected return. To see this, rewrite (16) as

$$R^N_{i,t+1} = \upsilon\frac{1+P^E}{P^N+\delta f(\Delta)}\frac{\eta\Gamma_{t+1}}{(1+\eta\Gamma_{t+1})^\alpha}\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}.$$

The properties of the gamma process imply that $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}$ is independent of Γ_{t+1}, and thus $R^N_{i,t+1}$ can be viewed as containing the component $\frac{\Gamma_{t+1}}{(1+\Gamma_{t+1})^\alpha}$, which is increasing in Γ_{t+1}, and the "idiosyncratic" component $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}$, which is independent of Γ_{t+1}. So, conventional reasoning would suggest that especially when an investor places a small weight of her wealth on new assets, the "idiosyncratic" component would become irrelevant, and the investor would value the hedging component of $R^N_{i,t+1}$, rendering its equilibrium excess return low, not high.

The resolution of this puzzle is tied with the fact that the idiosyncratic risk is priced in equilibrium, and increasingly so as Δ (and hence the allocation to new assets) becomes smaller. Indeed as Δ decreases, both the variance and the skewness of $\frac{1}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} dL_{j,t+1}$ become arbitrarily large, while the expected value is always one. In economic terms, this means that an investment in new assets results in a loss with probability approaching one, and results in a spectacularly high return with very small probability. Since the investor is risk averse, she finds this risk-return tradeoff unattractive and requires a higher compensation for that risk.

At the opposite extreme of perfect risk sharing ($\Delta = 1$, $\upsilon = 1$), the expected excess return of new ventures continue to exceed the expected excess of existing assets, but for an entirely different reason: in this case the marginal agent's consumption is perfectly correlated with aggregate consumption and new ventures deliver high payoffs (a large nunmber of blueprints) in states of the world where the consumption growth of existing investors would be high, not low.

In summary, for any constellation of parameters, the expected ratio of gross returns of new ventures to existing assets exceeds unity. Interestingly, this gap exists even in cases where the investor's marginal utility increases in Γ_{t+1}, i.e., in cases where the investor desires to hedge against fluctuations in displacement risk. Indeed, the smaller is Δ — and hence the higher the desire to hedge displacement risk — the larger is the gap in expected returns between the new venture portfolio and existing firms.

Inspection of (14) and (15) reveals that the returns R^E_{t+1} and R^N_{t+1} are negatively correlated. This result hinges critically on the fact that — purely for simplicity — the model includes only displacement shocks, but no "neutral" productivity shocks (or any other business cycle shocks for that matter), which would affect both new and existing firms in a similar fashion. Such shocks would easily render the correlation between R^E_{t+1} and R^N_{t+1} positive, but less than one; moreover they would introduce an additional source of risk for both existing and new assets, which would require additional compensation in form of an enhanced excess return.

4.4 Equilibrium interest rate

The equilibrium interest rate is declining in η, and is increasing in υ and Δ.

A higher value of η implies a more rapid (and more volatile and skewed) decline in the value of existing assets. Faced with such a modified profile, existing investors attempt to save more in order to leave their consumption profile unaffected. Since aggregate bond holdings must remain zero, the interest rate must decline to clear the market.

An increase in υ implies that existing agents can use purchases of new assets to reduce the impact of displacement shocks on their existing assets. Hence, less precautionary savings are required, and the interest rate increases. Finally, an increase in Δ also helps to reduce the idiosyncratic volatility of an investor's wealth, and hence helps reduce precautionary savings.

4.5 The participation arc Δ and the size of the financial industry

So far we have treated the participation arc Δ as an exogenous parameter. Now we discuss how it is determined inside the model. Equation (24) determines the size of the participation arc Δ, which is a monotone function of the resources devoted to the financial industry.

To gain some intuition on the determinants of the size of the financial industry, we provide the following comparative static results.

Lemma 5 *Δ is an increasing function of η and υ and a declining function of δ.*

Lemma 5 states, that as expected displacement increases (an increase in η), it becomes more meaningful to expend resources so as to reduce the uncertainty associated with risky new ventures. Similarly, the lower the fraction of shares that is retained by newly arriving agents (υ), the larger the incentive to expend resources to risk-share with the newly arriving agents. Finally, a higher value of δ increases the cost of the financial industry and hence the resources expended into it.

The following Lemma summarizes the impact of increased displacement on the economy, taking into account the impact of endogeneizing Δ.

Lemma 6 *Increased displacement (a higher η) implies a) a larger fraction of resources devoted to the financial industry (a higher Δ); b) a higher fraction of the aggregate portfolio directed towards new ventures (higher ϕ_N); c) a lower equilibrium interest rate r^f.*

5 Private equity and "real" assets

5.1 Private equity

To prepare for our discussion of private equity, we extend the model to allow some of the blueprints to accrue to existing firms. Specifically, we assume that the total number of new blueprints can be expressed as $\eta A_t \left(\Gamma_{t+1}^e + \Gamma_{t+1}\right)$, where Γ_{t+1}^e is the number of blueprints accruing to existing firms and Γ_{t+1} continues to capture the number of blueprints accruing to new firms. Γ_{t+1}^e is gamma distributed with shape a^E and rate b, so that $\Gamma_{t+1}^e + \Gamma_{t+1}$ is gamma distributed with shape $a^E + a$ and rate b.

How blueprints are allocated to existing firms is irrelevant for our purposes, since investors hold a value-weighted portfolio of all existing firms, so that the distribution of blueprints across existing fims is irrelevant.

In this version of the model, the return on existing firms becomes

$$R_{t+1}^{E}=\frac{A_t\pi_{t+1}\left(1+\eta\Gamma_{t+1}^{E}\right)\left(1+P^{E}\right)}{A_t\pi_t P^{E}}=\left(\frac{1+P^{E}}{P^{E}}\right)\frac{\left(1+\eta\Gamma_{t+1}^{E}\right)}{1+\eta\left(\Gamma_{t+1}^{E}+\Gamma_{t+1}\right)}\left(1+\eta\left(\Gamma_{t+1}^{E}+\Gamma_{t+1}\right)\right)^{1-\alpha},$$

while the return on new assets becomes

$$R_{t+1}^{N}=\frac{\upsilon A_t\left(\Pi_{t+1}+\pi_{t+1}\right)\frac{\eta}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}}d\Gamma_{i,t+1}}{A_t\pi_t\left(P^{N}+\delta f\left(\Delta\right)\right)}=\frac{R_{t+1}^{E}}{1+\eta\Gamma_{t+1}^{E}}\frac{P^{E}}{\pi^{N}+\delta f\left(\Delta\right)}\left(\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}}d\Gamma_{i,t+1}\right).$$

Repeating the proof of Proposition 1, it is straightforward to establish that all the results of Proposition 1 hold after replacing Z with

$$Z^{e}\equiv\frac{\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}}d\Gamma_{j,t+1}}{1+\eta\Gamma_{t+1}^{E}}.$$

Having introduced the possibility that some blueprints accrue to existing firms, we now sketch how to introduce private equity. We assume that each period a fraction of the existing firms lose their eligibility to receive new blueprints. Newly arriving entrepreneurs have the ability to purchase those firms from existing agents for a value $\Pi+P^{e}$ per blueprint, and restore their ability to receive an allocation of blueprints over the next period, at which point the firms are re-introduced into the public market.

The private equity firms can sell shares to the investors with whom they share the location. Mirroring the assumptions of the baseline model, these firms can only purchase companies located within a distance of $\frac{\Delta}{2}$ from the firm.

This version of the model would be equivalent to the baseline model. Hence, whether the blueprints arrive to newly created firms, or firms taken private and then re-listed is irrelevant for this model. The important economic force is the inability to hold a fully diversified portfolio of all non-publicly traded firms.

5.2 Real assets

So far we have studied a model where labor is the only factor of production. We now extend the model to introduce additional factors. We distinguish between two types of factors, namely those that are not tied to a specific blueprint but are useful for all productive purposes, and those that are specific to a given blueprint. An example of the first factor of production would be commercial real estate or commodities, while an example of the second factor of production would be specialized equipment required to manufacture a given intermediate good.

5.2.1. Land

The introduction of a factor such as land is straightforward. To be as explicit as possible that land is not tied to any intermediate good, we assume that land is useful only in the production of the final good. We discuss at the end of this section how to introduce land in the production of intermediate goods as well.

Land is owned by existing agents and rented out to final-good producing firms, so that aggregate output is given by

$$Y_t = F_t^{\zeta} \left(L_t^F\right)^{1-\alpha-\zeta} \left[\int_0^{A_t} x_{j,t}^{\alpha} dj\right], \tag{32}$$

where $\zeta \in (0, 1-\alpha)$ is the share of output that accrues to land. Total land is fixed and normalized to one. Given the Cobb-Douglas structure of (32), it follows that the rental rate of land is

$$r_t^F = \zeta Y_t. \tag{33}$$

Once again we construct an equilibrium where the price-to-rent ratio $P^F = \frac{P_t^F}{r_t^F}$ is constant, so that the return on land is given by $R_{t+1}^F = \frac{r_{t+1}^F + P_{t+1}^F}{P_t^F} = \frac{Y_{t+1}}{Y_t} \frac{1+P^F}{P^F}$. Repeating the arguments of Section 4.1, the wealth evolution of an individual investor is

$$\frac{W_{t+1}^i}{W_t^i} = \frac{1}{1-\lambda} \frac{W_{t+1}}{W_t} \left(\frac{\zeta\left(1+P^F\right)}{\alpha\left(1+P^E+P^N\right)+\zeta\left(1+P^F\right)} + \frac{\alpha\left(1-\alpha\right)\left(1+P^E\right)}{\alpha\left(1+P^E+P^N\right)+\zeta\left(1+P^F\right)} \left(\frac{1+\eta\upsilon\Gamma_{t+1}}{1+\eta\Gamma_{t+1}}\right) X_{i,t+1} \right), \tag{34}$$

for some new constants P^F, P^N, and P^E. Comparing (34) with (26), the only difference is that the wealth growth of an individual investor now gains a fraction $\frac{\zeta\left(1+P^F\right)}{\alpha(1+P^E+P^N)+\zeta(1+P^F)}$ of aggregate wealth growth. The reason is inutitive: Since land captures a constant fraction of total output, it actually benefits from higher values of Γ_{t+1}, since those are associated with higher output growth.

For sufficiently small Δ, υ, and ζ, it follows that the SDF is increasing in Γ_{t+1}. And since the return R^E_{t+1} is declining in Γ_{t+1}, while R^F_{t+1} is increasing in Γ_{t+1}, it follows that $E\left(R^F_{t+1}\right) < E\left(R^E_{t+1}\right)$. Indeed, in addition $E\left(R^F_{t+1}\right) < 1+r^f$, a result that, however, depends critically on the absence of neutral productivity shocks in the model.

5.2.2. Specialized equipment

Unlike a factor of production whose return is not tied to a specific blueprint, the behaviour of a factor of production that is closely tied with a specific blueprint would be quite different. To illustrate what would happen in such a case, we assume that in addition to labor, the production of the new goods requires a location-specific capital k_i

$$x_{i,t} = k_i^{\nu} l_{i,t}^{1-\nu},$$

where $l_{i,t}$ is the amount of labor used in the production of intermediate good i and k_i denotes an irreversible capital investment k_i that is specific to the location of a blueprint and its vintage. Capital for the arriving vintages is produced by converting consumption goods to capital goods with one unit of investment good requiring one unit of the consumption good.

Similar to the baseline model, the financial industry allows existing investors at time Δ to invest in capital goods in an arc of length Δ without knowing which locations will be receiving blueprints in the next period. This capital is then sold to the newly arriving firms in the respective locations once production commences at time $t+1$. We suspend the market for the sale of new firm shares to existing investors prior to the resolution of the uncertainty about their productivity; shares of new firms are tradeable only after their productivity is known. This assumption is inessential, but it will allow us to illustrate an analogy to the baseline model.

We will only sketch the solution of this version of the model, since the key intuitions are no different than the baseline model. In this version of the model aggregate output evolves according to

$$\frac{Y_{t+1}}{Y_t} = \left(\frac{A_{t+1}}{A_t}\right)^{1-(1-\nu)\alpha}$$

in steady state. The owners of capital goods extract a fraction ν of the present value of profits of the firms produced in location i, so that the return from investing in the new capital goods is given by

$$R^N_{t+1} = \frac{\nu A_t \left(\Pi_{t+1} + \pi_{t+1}\right) \frac{\eta}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}}{A_t k_t \left(1 + \psi f\left(\Delta\right)\right)} \tag{35}$$

which is very similar to equation (16), except that existing investors obtain a fraction ν of total profits and the cost of their investment is given by $A_t k_t \left(1 + \psi f\left(\Delta\right)\right)$, where $A_t k_t$ is the number of investment goods and $1 + \psi f\left(\Delta\right)$ is the cost per unit of capital good. Since the equilibrium of this model features a constant price-to-profits ratio in equilibrium, equation (35) can alternatively be expressed in q-theoretic fashion as

$$R_{t+1}^N = \frac{\nu \frac{\pi(k_{t+1})}{k_t} \left(1 + P^E\right) \frac{\eta}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t+1}}{1 + \psi f\left(\Delta\right)}, \tag{36}$$

where $\nu \frac{\pi(k_{t+1})}{k_t}$ is the marginal product of capital at firm inception and $1 + \psi f\left(\Delta\right)$ is the marginal cost of converting one unit of consumption to capital. Comparing (36) with (16) shows that ν plays a similar role to υ, in that it controls the fraction of new firm value that accrues to existing investors. However, unlike the baseline model the reason why existing investors capture that fraction is because they are providing the capital goods that are required for production, rather than insuring the entrepreneur.

In this version of the model the equilibrium quantity that adjusts to clear markets is k_{t+1}, the quantity of capital, rather than the valuation ratio P^N. Hence, all the intuitions of the baseline model that pertain to the magnitude of P^N carry over to the determination of k_{t+1}. For example, a mean–preserving spread in $\frac{\eta\nu}{\Delta} \int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{j,t+1}$, or an increase in ψ (a decrease in the efficiency of the financial industry) will reduce the investment in new capital goods in steady state. By contrast, such a spread would raise the attractiveness (and hence the equilibrium price) of a factor like land, which is not tied to a specific factor of production.

We conclude by noting that even though we drew a stark distinction between factors of production that are tied to specific blueprints and those that are not, in reality the distinction is not so stark, with many factors being convertible between different uses at some cost.

6 Further Discussion

We discuss here how to interpret in the context of our model various other economic quantities and institutions.

6.1 Labor income and pension funds

Labor income benefits from displacement in our model since the arriving firms compete for labor services, which are in fixed supply. Indeed, total wages increase at the same rate as aggregate output. Moreover, we have assumed that workers are hand-to-mouth consumers who are not actively participating in financial markets. These assumptions are purely for simplicity and can be easily relaxed.

Suppose for instance that workers are allowed to participate in financial markets. Moreover, suppose that the production of a unit of each intermediate good takes one unit of a labor composite good, which is a Cobb-Douglas aggregate of labor inputs provided by different cohorts of workers

$$l_t = \prod_{s=-\infty..t} (l_{t,s})^{a_{t,s}}, \tag{37}$$

where $l_{t,s}$ is the labor input of workers born at time s and the weights $a_{t,s}$ are given by $a_{t,s} = \frac{\Delta A_s}{A_t}$. This specification implies that even though the aggregate wage bill grows at the rate of aggregate output, the fraction of wages accruing to a given cohort of workers declines over time and indeed at the same rate as the profits of existing firms.

One motivation for such a specification is skill obsolecence: As the number of blueprints expands, the skills of a given cohort of workers becomes progressively less useful.

If one were to adopt equation (37), and allowed workers access to financial markets on the same terms as firm owners, then all our conclusions would carry through without modification: With such a specification, workers' human capital would exhibit a similar exposure to displacement to that of the value of existing firms, making workers eager to hedge displacement risk by investing in newly arriving companies.

In particular, if one took the view that pension funds invest on behalf of workers in a way that maximizes their welfare, then an increase in displacement activity would explain the increased popularity amongst pension funds of investment vehicles offering (positive) exposure to displacing firms.

6.2 Institutional investors and alternative asset classes

With the assumptions we made in the paper, existing investors differ in their return realizations, but collectively their portfolios finance a consumption process that is a declining fraction of aggregate consumption. Suppose instead that someone envisaged an institutional investor with the stated goal of financing a cash-flow stream that grows at the same rate as aggregate consumption (imagine, for instance, a pension fund or university endowment fund that needs to finance an expenditure stream growing at the same rate as aggregate output).

One commonly held view is that since profits are (roughly) a constant share of aggregate output, the replicating portfolio of such a stream would be un-levered equity: that is, invest a fraction of the portfolio in stocks and a fraction in bonds that corresponds to a firm's leverage ratio, so that the resulting portfolio has cash flows that match those of un-levered equity. Accordingly, these cash-flows would grow at the rate of aggregate output growth.

Figure 2 demonstrates clearly that such a strategy would not attain the stated goal. Our model offers a simple explanations for this fact, namely that, even though existing investors hold the market portfolio (i.e., all firms in existence) at any point in time, and all firms are un-levered, the consumption process that is financed by such a portfolio is a declining fraction of aggregate consumption. The reason is that a fraction of the portfolio is used up every period to finance new firm purchases. If one wanted to replicate a cash flow process that remains a constant (or, more generally, stationary) fraction of aggregate consumption, then one would have to invest disproportionately more (than the representative agent) in investment vehicles that are negatively exposed to displacement risk or in factors of production that are not as exposed to displacement.

This observation may help explain the attractiveness of alternative asset classes and real assets for institutional investors.

7 Calibration

In this section we calibrate the model. Our goal is to obtain a sense of the quantitative magnitudes for the share of the aggregate portfolio held in the form of alternative investments, the expected returns, interest rates, etc., that are implied by this model.

For calibration purposes it is convenient to maintain the assumption that agents have unit intertemporal elasticity of substitution, but have risk aversion γ that may be different than one. Moreover, our calibration focuses on the version of the model presented in Section 5.1, which allows existing firms to obtain blueprints. This is motivated by both realism and the desire to ensure that the dividends of existing firms and aggregate consumption growth are positively correlated (even in the absence of neutral productivity shocks).

The introduction of a risk-aversion coefficient $\gamma \neq 1$ doesn't change the model in a substantive way. The appendix describes the straightforward modifications necessary.

We abstract from issues such as illiquidity, lock-up periods, etc., which are quite common in private equity investments. As a result we choose to define a period to be equal to five years in our calibration. By lengthening the horizon, the Euler equation applies between times when the investor may actually have a chance to make a portfolio choice in reality. Consistent with common practice, we report returns at an annualized frequency to facilitate comparison with the literature.

Table 3 contains the parameters that we choose. The parameters a^E, a, and b control

a^E	4	β	0.97	υ	0.8
a	0.8	$1-\lambda$	0.98	Δ	0.5
b	0.6	γ	6	α	0.92
δ	0.01	$f(\Delta)$	1		

Table 3: Parameters used for the calibration

the distributions for the arrival of blueprints to existing and new firms. We choose these parameters to match as closely as possible the mean and variance of five-year real growth rates of aggregate consumption and the dividends of the market portfolio. To obtain a better understanding of these parameters, we note that the parameter a^E is five times larger than a. This means that (on average) the existing firms obtain five blueprints for every blueprint accruing to a newly born firm over a five-year horizon. As a result, the ratio $\frac{a^E}{a}$ controls how quickly the market value of existing firms declines as a fraction of total market capitalization over a five-year horizon. With these choices the market capitalization of existing firms is on average 87% of total market capitalization at the end of the five-year period, a number that is consistent with the data. The parameter b is mostly a scaling constant. With this choice of b we obtain an average (annualized) consumption growth rate of about 2 percent per year and a volatility of consumption equal to 1.5% (annualized). The volatility of dividend growth of the market portfolio is approximately 7% (annualized). The correlation between dividend growth of the market portfolio (and accordingly stock market returns in this model) and consumption growth is approximately 10%.

We choose $1-\lambda = 0.98$ to capture a birth rate of about 2% in the population, and $\beta = 0.97$ to match a real risk free rate of about 1%. The risk-aversion parameter $\gamma = 8$ is sufficient to match the (un-levered) equity premium. The parameter $\alpha = 0.92$ is chosen to match the profit share of output in aggregate data (about 8%). The quantity $\delta f(\Delta)$ captures the added value of the financial industry as a share of aggregate profits. Since the financial industry for the purposes of this paper is the private-equity industry, we choose a very low number. This number is actually quite inconsequential for any of the quantities of interest.

Finally, instead of fully specifying a function $f(\Delta)$ we choose Δ directly, since this is the quantity that enters returns, interest rates and the share of the market portfolio that is invested in alternative investments. A choice of Δ equal to 0.5 implies that a typical investor's portfolio of new firms has a correlation of $\sqrt{0.5}$ with the cross-sectional average of the returns of all private equity funds. Finally, the parameter $\upsilon = 0.8$ means that the

	$E(R^e)$	$E(R^N)$	r^f	ϕ_N
Baseline	3.54	13.40	1.76	6.48
$\Delta = 0.7$	2.08	5.39	3.34	8.48
$\upsilon = 0.5$	3.80	5.63	1.47	5.85
$\beta = 0.99$	3.47	13.10	-0.50	6.48
$\delta = 0.02$	3.55	13.43	2.04	6.48

Table 4: Un-levered excess return on existing assets ($E[R^e]$), un-levered excess return of the new assets portfolio provided by the intermediary ($E[R^N]$), risk-free rate (r^f), and fraction of assets invested in new assets (ϕ_N). All entries in the tables are percentages. All returns and excess returns are annualized. Rows correspond to different calibrations changing one parameter while keeping all others unchanged.

founder retains approximately 20% of a firm's equity.

Table 4 reports results of the calibration. The results are for un-levered returns. To relate un-levered to levered returns (which is what we observe in the data) one has to multiply the excess returns reported in the table by 1.6. Taking that into account, it is evident that the model can easily produce large equity premiums and also low real interest rates. Moreover, the fraction of assets invested in new assets is roughly consistent with the data.

The reason for the model's quantitative success is quite simple: the representative investor's wealth (and, accordingly, consumption) growth is more volatile than aggregate consumption. Indeed, an investor's (annualized) standard deviation of consumption growth is approximately 5.7% in this model. This larger standard deviation at the individual level is driven by the fact that innovation causes redistribution of wealth across investors with some benefiting and some losing. Even with perfect inter-cohort risk sharing ($\upsilon = 1$), the premiums remain substantial as long as intra-cohort risk sharing is imperfect.

References

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A Proofs

Proof of Proposition 1. A first-order condition for portfolio choice for an investor with logarithmic preferences is

$$\mathrm{E}\left[\frac{R^E_{t+1}-R^N_{t+1}}{\phi_B\left(1+r^f\right)+\phi_2 R^E_{t+1}+\phi_N R^N_{t+1}}\right]=0. \tag{38}$$

Using the definitions of ϕ_E and ϕ_N and imposing $\phi_B=0$ and market clearing in the stock markets implies $\phi_E=\frac{P^E}{P^E+P^N+\delta f(\Delta)}$ and $\phi_N=1-\phi_2$. Accordingly, using (14) and (16),

$$\phi_B\left(1+r^f\right)+\phi_E R^E_{t+1}+\phi_3 R^N_{t+1}=\frac{P^E}{P^E+P^N+\delta f\left(\Delta\right)}R^E_{t+1}\left(1+\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}}d\Gamma_{j,t+1}\right). \tag{39}$$

Using (39) and (16) inside (38) and noting that $\frac{P^E}{P^N+\delta f(\Delta)}=\frac{\phi_E}{1-\phi_E}$ leads to (17).

Having determined ϕ_E, it is straightforward to determine π^E and P^N. To start, we note that with logarithmic preferences $c=1-\beta\left(1-\lambda\right)$. Integrating (12), imposing asset market and goods market clearing $\frac{C_t}{A_t\pi_t}=\left(1-\delta f\left(\Delta\right)\right)$ and re-arranging yields

$$1+\pi^E+P^N=\frac{1-\delta f\left(\Delta\right)}{1-\beta\left(1-\lambda\right)}. \tag{40}$$

Combining (40) with $\phi_E=\frac{P^E}{P^E+P^N+\delta f(\Delta)}$ results in (21)–(22).

The requirement $\mathrm{E}\left[\frac{R^E_{t+1}-\left(1+r^f\right)}{\phi_1\left(1+r^f\right)+\phi_E R^E_{t+1}+\phi_N R^N_{t+1}}\right]=0$ together with $\phi_B=0$ yields (23). ■

Proof of Lemma 2. Aggregate wealth at time t is equal to the value of all existing firms plus the value of the newly arriving firms plus total profits, or:

$$W_t=A_t\pi_t\left(P^E+P^N+1\right). \tag{41}$$

Using (41) and (7), we arrive at (25). As for (26), it follows from (13), (39), (14) and the goods market clearing condition fact that $c\frac{W_t}{A_t\pi_t}+\delta f\left(\Delta\right)=1$, which implies

$$1-c=\frac{P^E+P^N+\delta f\left(\Delta\right))}{\left(1+\pi^E+P^N\right)}.$$

■

Proof of Lemma 3. When $\upsilon=1$ and $\Delta=1$, $\frac{M_{t+1}}{M_t}\propto\left(1+\eta\Gamma_{t+1}\right)^{\alpha-1}$, which is decreasing in Γ_{t+1}.

To prove the second sentence, suppose that $\upsilon<1$ or $\Delta<1$. Then differentiating $\frac{M_{t+1}}{M_t}$ with respect to Γ_{t+1} yields

$$\frac{\partial\left(\frac{M_{t+1}}{M_t}\right)}{\partial\Gamma_{t+1}}\propto\left(1+\eta\Gamma_{t+1}\right)^{\alpha}\left[\frac{\eta\alpha E\left(\frac{1}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L^{i,\Delta}_{t+1}}\right)}{\left(1+\eta\Gamma_{t+1}\right)}-E\left(\frac{\frac{\eta\upsilon}{\Delta}L^{i,\Delta}_{t+1}}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L^{i,\Delta}_{t+1}}\right)\right].$$

Fix a Δ. As υ becomes sufficienty small, the term inside square brackets converges to $\frac{\eta\alpha}{(1+\eta\Gamma_{t+1})}$, which is positive. Similarly, fix a value of υ and rewrite

$$\frac{\frac{\partial\left(\frac{M_{t+1}}{M_t}\right)}{\partial\Gamma_{t+1}}}{(1+\eta\Gamma_{t+1})^{\alpha}\frac{1}{\Gamma_{t+1}(1+\eta\Gamma_{t+1})}} \propto \eta\Gamma_{t+1}\alpha E\left(\frac{1}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}\right) - (1+\eta\Gamma_{t+1})\,E\left(\frac{\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}\right)$$
$$= \eta\Gamma_{t+1}\left[\alpha E\left(\frac{1}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}\right) - E\left(\frac{\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}\right)\right] - \mathrm{E}\left(\frac{\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}\right).$$

Since $\lim_{\Delta\to 0}\Pr\left(\frac{L_{t+1}^{i,\Delta}}{\Delta} > \kappa\right) = 0$ for any $\kappa > 0$, it follows that $\lim_{\Delta\to 0} E\left(\frac{1}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}\right) = 1$, $\lim_{\Delta\to 0} E\left(\frac{\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}{1+\frac{\eta\upsilon}{\Delta}\Gamma_{t+1}L_{t+1}^{i,\Delta}}\right) = 0$ and hence the sign of $\frac{\partial\left(\frac{M_{t+1}}{M_t}\right)}{\partial\Gamma_{t+1}}$ is positive. ■

Proof of Lemma 4. $\mathrm{E}\left[\frac{R^N}{R^E}\right]$ can be written as

$$\mathrm{E}\left[\frac{R^N}{R^E}\right] = \frac{\mathrm{E}\left[(1+Z)^{-1}\right]\mathrm{E}\left[Z\right]}{1-\mathrm{E}\left[(1+Z)^{-1}\right]} = \frac{\mathrm{E}\left[(1+Z)^{-1}\right]E\left[Z\right]}{\mathrm{E}\left[Z(1+Z)^{-1}\right]} = 1 - \frac{\mathrm{cov}\left(Z,(1+Z)^{-1}\right)}{\mathrm{E}\left[Z(1+Z)^{-1}\right]} > 1,$$

where the first equality follows from (17) and the last inequality follows from the fact that Z and $(1+Z)^{-1}$ are negatively correlated. ■

Proof of Lemma 5. It suffices to show that

$$\frac{\partial^2\mathrm{E}\left[\log\left(1+\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}\right)\right]}{\partial\Delta\partial\eta} > 0 \tag{42}$$

Differentiating $\mathrm{E}\left[\log\left(1+\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}\right)\right]$ with respect to η gives

$$\frac{\partial\mathrm{E}\left[\log\left(1+\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}\right)\right]}{\partial\eta} = \mathrm{E}\left[\frac{\frac{\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}}{1+\frac{\eta\upsilon}{\Delta}\int_{i-\frac{\Delta}{2}}^{i+\frac{\Delta}{2}} d\Gamma_{i,t+1}}\right].$$

Since $\frac{x}{1+\eta x}$ is a concave function of x it follows that for $\Delta_1 > \Delta_2$, $E\frac{\frac{1}{\Delta_1}\int_{i-\frac{\Delta_1}{2}}^{i+\frac{\Delta_1}{2}} d\Gamma_{i,t+1}}{1+\frac{\eta}{\Delta_1}\int_{i-\frac{\Delta_1}{2}}^{i+\frac{\Delta_1}{2}} d\Gamma_{i,t+1}} > E\frac{\frac{1}{\Delta_2}\int_{i-\frac{\Delta_2}{2}}^{i+\frac{\Delta_2}{2}} d\Gamma_{i,t+1}}{1+\frac{\eta}{\Delta_2}\int_{i-\frac{\Delta_2}{2}}^{i+\frac{\Delta_2}{2}} d\Gamma_{i,t+1}}$, which proves (42). ■