Markov-Switching Mixed-Frequency VAR Models*

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Abstract

This paper introduces regime switching parameters in the Mixed-Frequency VAR model. We first discuss estimation and inference for Markov-switching Mixed-Frequency VAR (MSMF-VAR) models. Next, we assess the finite sample performance of the technique in Monte-Carlo experiments. Finally, the MSMF-VAR model is applied to predict GDP growth and business cycle turning points in the euro area. Its performance is compared with that of a number of competing models, including linear and regime switching mixed data sampling (MIDAS) models. The results suggest that MSMF-VAR models are particularly useful to estimate the status of economic activity.

Keywords: Markov-switching, MIDAS, Mixed-frequency VAR, Nowcasting, Forecasting.

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1 Introduction

Macroeconomic variables are sampled at different frequencies. In empirical studies, the econometrician has thereby to choose the sampling frequency of the model. The first solution consists in aggregating all the high-frequency variables to the frequency of the lowest frequency variable. This is a common choice in VAR studies. Another solution is to interpolate the low-frequency variable to the frequency of the high-frequency variable, which is usually done with the Kalman filter (see e.g., Aruoba et al. (2009), Giannone et al. (2008), Mariano and Murasawa (2003), Frale et al. (2011), and Foroni et al. (2013)). Kuzin et al. (2011) and Bai et al. (2013) use the Kalman filter for VAR forecasting with mixed-frequency data. See also Foroni et al. (2013) for an overview. Ghysels (2012) recently introduced a mixed-frequency VAR model where the vector of dependent variables includes both the low-frequency variable and the high-frequency variables, the latter stacked depending on the timing of the release. The advantage of this approach is to avoid the estimation of a mixed-frequency VAR model via the Kalman filter that often proves to be computationally cumbersome. Alternatively, in a Bayesian context, Chiu et al. (2011) developed a Gibbs sampler that allows to estimate VAR models with irregular and mixed-frequency data, see also Schorfheide and Song (2012). Finally, one can work directly with mixed-frequency data. This is the approach that is used by MIDAS specifications where the aggregation of the high-frequency variable is carried out in a parsimonious and data-driven manner (see e.g., Ghysels et al. (2004), Andreou et al. (2013) and Francis et al. (2011)). Banbura et al. (2012) and Foroni and Marcellino (2013c) provide overviews on the use of mixed-frequency data in econometric models.

An increasing number of works specifically evaluates the effects of the choice of the sampling frequency in a macroeconomic forecasting context (starting with Clements and Galvao (2008), see also e.g. Foroni and Marcellino (2013a), Foroni et al. (2012), Banbura et al. (2012)) but also in structural studies (Foroni and Marcellino (2013b) and Kim (2011)). However, relatively few studies consider the issue of time variation in models dealing with mixed-frequency data (see e.g., Galvão (2013), Guérin and Marcellino (2013), Carriero et al. (2013), Marcellino et al. (2013) and Camacho et al. (2012)), and most of these few studies only consider the univariate case.

In this paper, we allow for regime switching parameters in the mixed-frequency VAR model, introducing the Markov-Switching Mixed-Frequency VAR model (MSMF-VAR). The relevance of Markov-Switching models in econometrics is now well established and a large number of studies have been published after the seminal paper by Hamilton (1989), see in particular Krolzig (1997) in a VAR context, but all these studies are based on same frequency variables. The MSMF-VAR model permits us to explicitly model time variation in the relationship between the high- and low-frequency variables. As a by-product of our estimation, we also obtain high-frequency estimates of the low-frequency variable(s) in the case of the mixed-frequency VAR model estimated via the Kalman filter. For example, in a system with quarterly GDP and monthly indicators, we can obtain monthly estimates of GDP, which are of direct interest. Modeling time variation through Markov-Switching models is also attractive since it allows us to endogenously estimate and forecast the probabilities of being in a given regime.

We introduce regime switching parameters in the mixed-frequency VAR model estimated via the Kalman filter as well as in the mixed-frequency VAR model of Ghysels (2012) that does not require to estimate a state-space model and thereby permits a straightforward estimation. We compare these new models with the Markov-Switching MIDAS model of Guérin and Marcellino (2013), and discuss the distinctive features of these models.

The paper is structured as follows. The second section introduces the Markov-Switching Mixed-frequency VAR models and discusses their use for forecasting the variables of interest and the underlying unobservable states. The third section evaluates in a Monte-Carlo experiment the forecasting performance of the MSMF-VAR models compared with a number of alternative forecasting models. The fourth section presents an empirical application for the prediction of GDP growth and business cycle turning points in the euro area. Section 5 concludes.

2 Modeling and Forecasting with Markov-Switching Mixed-Frequency models

2.1 Markov-Switching Mixed-Frequency VAR

In this subsection, we introduce two alternative Markov-Switching mixed-frequency VAR (MSMF-VAR) models. First, we present the MSMF-VAR model cast in state-space form so as to accommodate the different frequency mixes. This model is labeled MSMF-VAR (KF), where KF stands for Kalman filter. Second, we introduce regime switching parameters in the mixed-frequency VAR model recently proposed by Ghysels (2012) where mixed frequency data are analyzed through a stacked-vector system. This model is labeled MSMF-VAR (SV), where SV stands for stacked-version.

2.1.1 Mixed-frequency data modeled via a state-space representation

For the sake of clarity, we illustrate the model using a single low-frequency variable, sampled at the quarterly frequency, and a single high-frequency variable, sampled at the monthly frequency. This corresponds to the standard macroeconomic forecasting context where one wants to forecast a quarterly variable, say GDP growth, with a monthly indicator, such as industrial production, an interest rate spread, or a survey on economic conditions. Models with more low- or high-frequency variables can be specified along the same lines.

A key feature of the Mixed-Frequency VAR (MF-VAR) model is to work at the frequency of the high-frequency variable. Quarterly GDP is thus disaggregated at the monthly frequency using the geometric mean:

$$Y_t = 3(Y_t^* Y_{t-\frac{1}{3}}^* Y_{t-\frac{2}{3}}^*)^{\frac{1}{3}}, \tag{1}$$

where Y_t is the observed quarterly GDP in quarter t and Y^*_{t-h} is the unobserved monthly GDP of the months $h = \{0, \frac{1}{3}, \frac{2}{3}\}$ belonging to quarter t. Taking logarithms and after some transformation, this yields:

$$y_t = \frac{1}{3}y^*_t + \frac{2}{3}y^*_{t-\frac{1}{3}} + y^*_{t-\frac{2}{3}} + \frac{2}{3}y^*_{t-1} + \frac{1}{3}y^*_{t-\frac{4}{3}}, \tag{2}$$

where y_t is the observed quarterly growth rate in quarter t and y^*_{t-h} is the monthly growth rate of GDP in months $t-h$.

Note that we use the geometric mean rather than the arithmetic mean. Using the arithmetic mean rather than the geometric mean would involve the estimation of a non-linear state space model, which would complicate the estimation. Besides, the approximation involved by the use of the geometric mean is typically small when dealing with GDP growth in developed economies. Finally, it is the approach commonly used in the literature, see e.g. Mariano and Murasawa (2003).[1]

The state vector of unobserved variables $\mathbf{s_t}$ includes the monthly changes in GDP growth y^*_{t-h} as well as the high frequency indicator x_{t-h}. It is defined as follows:

$$\mathbf{s}_t = \begin{bmatrix} z_t \\ \cdot \\ \cdot \\ \cdot \\ z_{t-\frac{4}{3}} \end{bmatrix},$$

where:

$$\mathbf{z}_{t-h} = \begin{bmatrix} y^*_{t-h} \\ x_{t-h} \end{bmatrix}.$$

The state-space representation of the MF-VAR(p) is described by the following transition and measurement equations:

$$\mathbf{s}_{t+h+\frac{1}{3}} = \mathbf{A}\mathbf{s}_{t+h} + \mathbf{B}\mathbf{v}_{t+h+\frac{1}{3}}, \tag{3}$$

$$\begin{bmatrix} y_{t+h} \\ x_{t+h} \end{bmatrix} = \begin{bmatrix} \mu_y \\ \mu_x \end{bmatrix} + \mathbf{C}\mathbf{s}_{t+h}, \tag{4}$$

where $\mathbf{v}_{t+h} \sim N(\mathbf{0}, I_2)$ and the system matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are given by:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A_1} \\ \mathbf{A_2} \end{bmatrix},$$

$$\mathbf{A_1} = [\mathbf{\Phi_1} ... \mathbf{\Phi_p} \;\; \mathbf{0_{2*2(5-p)}}],$$

[1]Note that one could also handle missing observations by varying the dimension of the vector of observables as a function of time t (see e.g., Schorfheide and Song (2012)).

$$\mathbf{A_2} = [\mathbf{I_8} \;\; \mathbf{0_{8*2}}],$$

$$\mathbf{B} = \begin{bmatrix} \Sigma^{1/2} \\ 0_{8*2} \end{bmatrix},$$

and,

$$\mathbf{C} = [\mathbf{H_0} ... \mathbf{H_4}],$$

where the matrix $\mathbf{C}$ contains the coefficient matrices in the lag polynomial $H(L_m) = \sum_{i=0}^{4} H_i L_m^i$, which is defined according to the aggregation constraint defined in equation (2):

$$\mathbf{H}(L_m) = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{pmatrix} L_m + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} L_m^2 + \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & 0 \end{pmatrix} L_m^3 + \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 0 \end{pmatrix} L_m^4.$$

Note that this specification of the MF-VAR differs from the one reported in Camacho and Perez-Quiros (2010) and Bai et al. (2013), where the low-frequency variable is related to the high-frequency indicators through common factors (i.e., they use a mixed-frequency factor model). By contrast, the specification we consider is similar to the one in Zadrozny (1990) and Kuzin et al. (2011).

We now include regime switches in potentially all the parameters of the MF-VAR(p) model: the intercepts, the autoregressive parameters as well as the variance of the innovations.[2] The state-space representation of the Markov-Switching MF-VAR (MSMF-VAR) is then given by:

$$\mathbf{s}_{t+h+\frac{1}{3}} = \mathbf{A}(S_{t+h+\frac{1}{3}})\mathbf{s}_{t+h} + \mathbf{B}(S_{t+h+\frac{1}{3}})\mathbf{v}_{t+h}, \tag{5}$$

$$\begin{bmatrix} y_{t+h} \\ x_{t+h} \end{bmatrix} = \begin{bmatrix} \mu_y(S_{t+h}) \\ \mu_x(S_{t+h}) \end{bmatrix} + \mathbf{C}\mathbf{s}_{t+h}, \tag{6}$$

where S_t is an ergodic and irreducible Markov-chain with a finite number of states $S_t = \{1, ..., M\}$ defined by the following constant transition probabilities:

$$p_{ij} = Pr(S_{t+h} = j | S_{t+h-\frac{1}{3}} = i), \tag{7}$$

$$\sum_{j=1}^{M} p_{ij} = 1 \quad \forall i, j \epsilon \{(1, ..., M)\}. \tag{8}$$

Following Mariano and Murasawa (2003), we replace the missing observations with zeros, and we rewrite the measurement equation in such a way that the Kalman filter skips the missing observations.

The model is estimated with the Kalman filter combined with the Hamilton filter for Markov-Switching models (see Appendix A for details). We also need to make an approximation at the end of the Kalman and Hamilton filters to avoid the proliferation of cases

[2]Note that Camacho (2013) also considers regime switches in a mixed-frequency VAR similar to ours. However, he concentrates on the in-sample performance of the MSMF-VAR model for estimating business cycle turning points in the US and does not perform a forecasting exercise.

to be considered as in Kim and Nelson (1999). Appendix A reports the equations of the Kalman filter as well as the Kim and Nelson (1999) approximation necessary for state-space models with regime switches. The model parameters are estimated by maximum likelihood using the Expectation Maximization (EM) algorithm. The expectation step (”E” step) applies the Kalman and Hamilton filter to obtain an estimate of the log-likelihood, and the maximization step (”M” step) maximizes the log-likelihood obtained from the expectation step using the BFGS algorithm from the OPTMUM library of the Gauss software. We iterate over the ”E” and ”M” steps until the algorithm has converged.

2.1.2 Mixed-frequency data modeled via a stacked-vector system

We present here the regime switching version of the mixed-frequency VAR model of Ghysels (2012). In this case, the high-frequency variable is stacked in the vector of dependent variables depending on the timing of the release of the high frequency data. This permits to avoid the estimation via the Kalman filter that often proves to be computationally demanding. As in the previous subsection, we describe the model with a single low-frequency variable and a single high-frequency variable to facilitate the notation. In particular, we illustrate the model in the case where one wants to forecast one quarterly variable using one monthly variable that is released three times a quarter. However, extensions to larger systems involving more low- or high-frequency variables naturally follow. Alternatively, a model considering more than two different sampling frequencies could be specified along the same lines. The crucial point of this type of mixed-frequency VAR model is to stack observations from the high-frequency variable x_t in the vector of dependent variables depending on the timing of the release of the high-frequency indicator.

Denote as y_t the quarterly variable, $x_{t-\frac{2}{3}}, x_{t-\frac{1}{3}}, x_t$ the release of the monthly indicator in the first, second and third month of quarter t, respectively. This class of mixed-frequency VAR model can therefore be written as follows:

$$z_t = A_0 + \sum_{j=1}^{p} A_j z_{t-j} + u_t, \tag{9}$$

where $z_t = (x_{t-\frac{2}{3}}, x_{t-\frac{1}{3}}, x_t, y_t)'$ is the vector of observable time series variable, A_0 is the vector of intercepts, the A_j's $(j = 1, ..., p)$ are the coefficient matrices and u_t is the white noise error term with mean zero and positive definite variance covariance matrix Σ_u, that is $u_t \sim (0, \Sigma_u)$.

Regime switching can be included in possibly all parameters of the model: intercepts, autoregressive coefficient matrices and the variance-covariance matrix of the error term, so that the MSMF-VAR (SV) model can be specified as follows:

$$z_t = A_0(S_t) + \sum_{j=1}^{p} A_j(S_t) z_{t-j} + u_t(S_t), \tag{10}$$

where S_t is an ergodic and irreducible Markov-chain with a finite number of states as defined in equations (7) and (8). The conditional distribution of u_t given S_t is assumed to be normal $u_t|S_t \overset{i}{\sim} N(0, \Sigma_u(S_t))$.

In the Monte-Carlo experiments and the empirical application, we consider two different versions for the MSMF-VAR (SV) model. The first version assumes no restrictions on the coefficient matrices A_j, this model is therefore denoted as MSMF-VAR(SV-U). The second version instead assumes that the coefficient matrices have particular structures. In detail, we assume that the high-frequency indicator follows an ARX(1) process, following an example in Ghysels (2012). More details about the restrictions we impose are reported in Appendix B. The restricted version of the MSMF-VAR (SV) model is therefore denoted as MSMF-VAR(SV-R).

The estimation of this type of MSMF-VAR is similar to the estimation of a single frequency MS-VAR and it is done in a classical maximum likelihood framework, as discussed in Ghysels (2012) for the linear case.[3] We therefore use the EM algorithm without requiring the Kalman filter, as in Krolzig (1997).

2.2 Markov-Switching MIDAS model

Markov-Switching parameters were first introduced in MIDAS models by Guérin and Marcellino (2013). The Markov-Switching MIDAS model can be written as follows:

$$y_t = \beta_0(S_t) + \beta_1(S_t)B(L^{1/m};\theta(S_t))x_{t-h}^{(m)} + \epsilon_t(S_t), \tag{11}$$

where $B(L^{1/m};\theta(S_t)) = \sum_{j=1}^{K} b(j;\theta(S_t))L^{(j-1)/m}$, $L^{s/m}x_{t-1}^{(m)} = x_{t-1-s/m}^{(m)}$ and $\epsilon_t|S_t \overset{i}{\sim} N(0,\sigma^2(S_t))$.

The MIDAS weight function $b(j;\theta(S_t))$ aggregates the high-frequency variable to the frequency of the dependent variable y_t through a parametric weight function. The MIDAS parameters θ govern the aggregation process so that one can include a large number of lags for the high frequency variable with only a limited number of parameters. A key feature of the MIDAS aggregation scheme is therefore to aggregate the high frequency variable in a parsimonious way.

Two common choices for the weight function $b(j;\theta(S_t))$ (see e.g. Ghysels et al. (2007)) are the exponential Almon lag specification and the beta weight function. A standard choice in the literature is to use two parameters for the MIDAS weight function (i.e., $\theta = \{\theta_1, \theta_2\}$). The exponential Almon lag specification and the beta weight function can then be written as follows:

$$b(j;\theta(S_t)) = \frac{exp(\theta_1(S_t)j + \theta_2(S_t)j^2)}{\sum_{K=1}^{K} exp(\theta_1(S_t)j + \theta_2(S_t)j^2)}, \tag{12}$$

[3] Chauvet et al. (2013) study the link between economic fluctuations and financial markets using the MF-VAR(SV-R) model estimated via seemingly unrelated regressions (see e.g., Greene (2011)). However, their specification of the MF-VAR(SV-R) differs from ours in that they assume that the indicator follows an ARX(1) process but they also make use of MIDAS restrictions for the equation describing the relation between the low-frequency and high-frequency variables. Also, they consider a mix of monthly and daily data so that the dimension of their models is very large. As a result, they implement a two-step estimation procedure to facilitate the estimation. In our case, however, we do not impose MIDAS restrictions and consider a relatively small system (i.e., a mix of quarterly and monthly data). Maximum likelihood estimation therefore remains fairly straightforward.

$$b(j;\theta(S_t)) = \frac{(d/D)^{\theta_1(S_t)} - (1-d/D)^{\theta_2(S_t)-1}}{\sum_{j=0}^{K}(j/D)^{\theta_1(S_t)} - (1-j/D)^{\theta_2(S_t)-1}}. \tag{13}$$

As in the VAR case, the regime generating process is an ergodic Markov-chain with a finite number of states $S_t = \{1, ..., M\}$ defined by the following constant transition probabilities:

$$p_{ij} = Pr(S_{t+1} = j | S_t = i), \tag{14}$$

$$\sum_{j=1}^{M} p_{ij} = 1 \forall i, j \epsilon \{1, ..., M\}. \tag{15}$$

ADL-MIDAS type of specifications introduce autoregressive dynamics in MIDAS models in a straightforward manner (see e.g. Andreou et al. (2013)) so that the MSADL-MIDAS model can be written as follows:

$$y_t = \beta_0(S_t) + \beta_1(S_t)B(L^{1/m}; \theta(S_t))x_{t-h}^{(m)} + \lambda(S_t)y_{t-d} + \epsilon_t(S_t). \tag{16}$$

In the rest of the paper we will use the MIDAS and ADL-MIDAS with exponential Almon lag specification.

2.3 Comparison between MS-MIDAS and MSMF-VAR models

The main differences between MS-MIDAS and MSMF-VAR models are the following.

First, MIDAS models make direct forecasts whereas forecasts from MF-VAR models (both state-space and stacked vector system versions) are usually calculated as iterated forecasts. A comparison between direct and iterated forecasts can be found, e.g., in Chevillon and Hendry (2005) and Marcellino et al. (2006).

Second, MIDAS models do not model the behavior of the high-frequency variable whereas MF-VAR models explicitly model the dynamics of the high frequency variable. As a result, provided that MF-VAR models are well specified, one can expect to achieve more precise forecasts with respect to MIDAS models. However, MIDAS models are more parsimonious and are thus less prone to misspecifications than MF-VAR models. In particular, the MF-VAR model version from Ghysels (2012) is typically subject to parameter proliferation, especially in systems with different frequency mixes and a large number of lags.

Third, the estimates and forecasts for the regime probabilities from MS-MIDAS models are made at the quarterly frequency, albeit they can be updated on a monthly basis. The MSMF-VAR (SV) models also estimate the regime probabilities at the quarterly frequency. By contrast, MSMF-VAR models estimate and forecast the regime probabilities directly at the monthly frequency.

Fourth, the MSMF-VAR (KF) model permits to obtain a monthly estimate of the quarterly variable, which is often of interest by itself.

Finally, by construction the regime predictions from the MS-MIDAS and MSMF-VAR (SV) models are necessarily lower than the last estimate of the regime probabilities, and do not use the most timely information from the monthly indicator. MSMF-VAR (KF) models instead run a combination of the Kalman and Hamilton filter forward for computing the regime forecasts so that MSMF-VAR (KF) models include the latest information from the high-frequency indicator to predict the regime probabilities.

3 Monte-Carlo experiments

3.1 In-sample estimates

Our first Monte-Carlo experiment is designed to check how well our estimation algorithm performs in finite samples.

First, we assume that the data are generated by a regime-switching VAR(1) process. The population model is defined as follows:

$$\begin{pmatrix} y_t \\ x_t \end{pmatrix} = \begin{pmatrix} \mu_1(S_t) \\ \mu_2(S_t) \end{pmatrix} + \begin{pmatrix} 0.5 & \alpha \\ 0.1 & 0.5 \end{pmatrix} \begin{pmatrix} y_{t-1} \\ x_{t-1} \end{pmatrix} + \epsilon_t(S_t)$$

where

$$\begin{aligned} \mu_1(S_t) &= \{-1, 1\}, \\ \mu_2(S_t) &= \{-1, 1\}, \\ \epsilon_t|(S_t &= 1) \sim N(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}), \\ \epsilon_t|(S_t &= 2) \sim N(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 0.4 \\ 0.4 & 2 \end{pmatrix}), \\ \alpha &= \{0.1, 0.5\}. \end{aligned}$$

The parameter α therefore governs the degree to which the first dependent variable is affected by the second dependent variable. Note that, unlike the model generally discussed in section 2, we only consider here switches in the intercepts and in the variance-covariance matrix of the shocks, but not in the autoregressive matrix. There are no conceptual issues in terms of estimation which prevent us from having switches in the autoregressive term. We consider changes in the intercepts because they are the most common sources of forecast failures (see, e.g, Clements and Hendry (2001)). Moreover, considering regime changes in the autoregressive coefficient matrix for all the regime switching models we analyze would substantially complicate the estimation by increasing computational time and convergence problems of the algorithm.

We also consider differences in the regimes' duration. The first set of transition probabilities is defined as follows:

$$(p_{11}, p_{22}) = (.95, .95).$$

The second set of transition probabilities instead implies that the first regime has a lower duration than the second regime:

$$(p_{11}, p_{22}) = (.85, .95).$$

Second, we assume that the data are generated by two distinct DGPs, one for the low-frequency variable y_t and one for the high-frequency variable x_t, that is we consider a mixed-frequency DGP. The population models for y_t and x_t are autoregressive models defined as follows:[4]

$$y_t = \mu_1(S_t) + 0.5y_{t-1} + \epsilon_t^y(S_t)$$

$$x_t = \mu_2(S_t) + 0.5x_{t-1} + \epsilon_t^x(S_t)$$

where $\mu_1(S_t) = \{-1, 1\}$, $\mu_2(S_t) = \{-1, 1\}$,

$$\begin{aligned} \epsilon_t^y|(S_t = 1) &\sim N(0,1), \quad & \epsilon_t^x|(S_t = 1) &\sim N(0,1) \\ \epsilon_t^y|(S_t = 2) &\sim N(0,2), \quad & \epsilon_t^x|(S_t = 2) &\sim N(0,2) \end{aligned}$$

For each DGP, the sample size expressed at the frequency of the low frequency variable is $T = 250$. As we consider the standard case of a mix between quarterly and monthly variables, time series with length $T * 3$ were initially generated so that the sample size expressed at the frequency of the high frequency variable includes 750 observations.[5]

We compare the in-sample fit of the models under scrutiny calculating the in-sample mean squared error defined as follows:

$$IS - MSE = \sum_{t=1}^{T} (\hat{y}_t - y_t)^2 / T. \tag{17}$$

The MSE is computed for each replication $n = \{1, .., N\}$, with $N = 1000$ being the number of Monte-Carlo replications, and then averaged across replications.

Quadratic Probability Score (QPS) and Log Probability Score (LPS) can also be calculated for assessing the ability of non-linear models to detect turning points. QPS and LPS are defined as follows:

$$QPS = \frac{2}{T} \sum_{t=1}^{T} (P(S_t = 1) - S_t)^2, \tag{18}$$

$$LPS = -\frac{1}{T} \sum_{t=1}^{T} (1 - S_t) log(1 - P(S_t = 1)) + S_t log(P(S_t = 1)), \tag{19}$$

where S_t is a dummy variable that takes on a value of 1 if the true regime is the first regime in period t, and $P(S_t = 1)$ is the smoothed probability of being in the first regime in period t. The median across replications is then reported.

[4]Note that we assume that the same Markov chain S_t governs the parameter changes for both DGPs. As such, this implies that both variables are driven by the same comovements, which is a common modelling choice in business cycle analysis (see e.g., Chauvet (1998)).

[5]Note that for the mixed-frequency DGPs, time series with length $T = 250$ observations were generated for y_t and $T = 750$ for x_t. Also, for each draw, we discarded the first 500 observations to account for start-up effects.

In this exercise, we consider fifteen different models, a list similar to that used in the empirical application in section 4. Specifically, we have AR(1), ADL(1,1), linear MIDAS, linear ADL-MIDAS, MS-MIDAS, MSADL-MIDAS, univariate MS model, linear VAR, regime-switching VAR, linear mixed-frequency VAR (both the state-space and the stacked-vector system versions) and regime switching mixed-frequency VAR models (both the state-space and the stacked-vector system versions). For the SV models, we consider both the restricted (R) and unrestricted (U) versions. Appendix B provides more details about each model under scrutiny.

Table 1 presents the results for the Monte Carlo experiments. Panel A and Panel C report the results for the DGP with equal transition probabilities for the single-frequency DGP and the mixed-frequency DGP, respectively. Panel B and Panel D show the results for the DGP that implies a lower duration for regime 2 compared to regime 1 for both types of DGP. For panels A and B, we present results with the two different parameter values for α (i.e., $\alpha = 0.1$ and $\alpha = 0.5$).

Several main findings emerge from Table 1. First, for the single-frequency DGP, across the four different cases, the MSMF-VAR model estimated via the Kalman filter (i.e., the MSMF-VAR(KF) model) obtains the best results in terms of QPS and LPS. In other words, our simulation results indicate that - based on these DGPs - the MSMF-VAR (KF) model yields the best estimates of the in-sample regime probabilities. Second, the MSMF-VAR(SV-R) model yields the best in-sample estimates across all four DGPs, with the MS-MIDAS a very close second best for $\alpha = 0.1$ and the MSADL-MIDAS model a very close second best for $\alpha = 0.5$. Third, in the case of a mixed-frequency DGP, the univariate MS-AR model performs best followed by the MSADL-MIDAS model, which is not surprising given that the true model for y_t is generated from a univariate model. Fourth, also in the case of the mixed-frequency DGP, the best estimate for the regime probabilities are obtained by the single-frequency MS-VAR model and it is closely followed by the MSMF-VAR (SV-R) model. As such, this can be explained by the fact that since both y_t and x_t are generated from the same Markov-chain, using both variables in a model helps for the inference on regimes. Also, note that conditional on these mixed-frequency DGPs, the MSMF-VAR(KF) model performs relatively poorly for estimating the regimes compared with the other models. Finally, it is also worth mentioning that linear models fit the data much worse than MS specifications. As we will see, this finding no longer generally holds in an out-of-sample context.

3.2 Out-of-sample estimates

We now assess the finite sample forecasting performance for the low frequency variable of the Markov-Switching Mixed-Frequency VARs, comparing it with a number of competing forecasting models in a controlled setup.

The design of this Monte Carlo experiment is the following. For each draw, we set the evaluation sample to $T_{eval} = 50$ and we calculate recursively one-step-ahead forecasts until we reach the end of the full estimation sample. When calculating one-step-ahead forecasts, we make sure that all forecasting models use the same information set. Therefore, when we forecast the next quarter we do not use any monthly information on that quarter. As a result, the design of our Monte-Carlo experiment is relatively unfavorable to mixed-frequency data models in that this class of models could explicitly accommodate more timely information. Also, to reduce the computational time, we fix the parameter estimates at the values obtained with the first estimation sample, rather than recursively updating the values. Limited evaluation of fully recursive estimation has produced similar results.

The one-step-ahead forecasts are generated from the fifteen forecasting models described in the previous subsection. Forecasts for regime switching models are calculated in the standard way by weighting the conditional forecasts upon each regime by the predicted probabilities of being in a given regime. In the case of two regimes, one-step-ahead forecasts are calculated as:

$$\hat{y}_{t+1|t} = \hat{y}_{t+1|t,1} P(S_{t+1|t} = 1) + \hat{y}_{t+1|t,2} P(S_{t+1|t} = 2) \tag{20}$$

where:

- $\hat{y}_{t+1|t,i}$ is the one-step-ahead forecast for the dependent variable y_t conditional on the process being in state i at time $t+1$ given the past observable information.
- $P(S_{t+1|t} = i)$ is the conditional probability of the process being in state i at time $t+1$ given the past observable information.

Generalizing to h-step-ahead forecasts and M regimes, equation (20) becomes:

$$\hat{y}_{t+h|t} = \sum_{i=1}^{M} \hat{y}_{t+h|t,i} P(S_{t+h|t} = i). \tag{21}$$

We compare the forecasting performance for the levels of the low-frequency variable using the Mean Squared Forecast Error (MSFE). When using a regime switching model, we also calculate the in-sample Quadratic Probability Score (QPS) and Log Probability Score (LPS) to assess the ability of non-linear models to predict turning points in a real-time context. QPS and LPS are then defined as follows:

$$QPS = \frac{1}{T_{eval}} \sum_{i=1}^{T_{eval}} \Big(\frac{2}{T_i} \sum_{t=1}^{T_i} (P(S_t = 1) - S_t)^2\Big), \tag{22}$$

$$LPS = \frac{1}{T_{eval}} \sum_{i=1}^{T_{eval}} \Big(-\frac{1}{T_i} \sum_{t=1}^{T_i} (1 - S_t) log(1 - P(S_t = 1)) + S_t log(P(S_t = 1))\Big), \tag{23}$$

where S_t is a dummy variable that takes on a value of 1 if the true regime is the first regime in period t, and $P(S_t = 1)$ is the smoothed regime probability of being in the first regime in period t. T_i is the size of the sample for evaluation period i of the estimation sample and T_{eval} is the size of the evaluation sample.

Table 2 reports results with $N = 1000$ Monte Carlo replications. Conditional on the single-frequency DGP with equal transition probabilities and $\alpha = 0.1$, the best forecasting model is the MSMF-VAR(SV-U) model (Panel A). The MSMF-VAR(SV-U) also obtains the best forecasting results with $\alpha = 0.5$. Besides, conditional on the single-frequency DGP with transition probabilities $p_{11} = 0.85$ and $p_{22} = 0.95$ with $\alpha = 0.1$, the MS-VAR model yields the best continuous forecasts, while for $\alpha = 0.5$ the best model in terms of continuous forecasts is again the MSMF-VAR (SV-U) model. For the mixed-frequency DGPs with equal transition probabilities, the best forecasting model is the MS-VAR model and it is closely followed by the ADL and ADL-MIDAS models. The stacked-vector system version of the MF-VAR instead perform best for the mixed-frequency DGP with different transition probabilities. Table 2 also shows the QPS and LPS for each model across the four DGPs. The figures indicate that the MSMF-VAR (KF) model yields systematically the best discrete estimates of the regimes with the single-frequency DGP (Panels A and B). Instead, the MSMF-VAR (SV) models tend to obtain the best estimates of the regimes with the mixed-frequency DGP (Panels C and D).

Overall, the results of the Monte Carlo experiments suggest that our estimation procedure for MSMF-VARs performs well also in finite samples. Moreover, not accounting for MS features deteriorates the in- and out-of-sample performance of all models, the former more than the latter.

4 Empirical application

4.1 Data

The empirical application focuses on euro area macroeconomic variables. In particular, we use the euro area GDP growth as quarterly variable, and a set of four monthly indicators often considered as good predictor for growth: the Economic Sentiment Indicator (ESI), the M1 monetary aggregate, headline industrial production and the slope of the yield curve, see Table 3. We use real-time data obtained from the ECB real time database (see Giannone et al. (2012) for a description of the database) except for the slope of the yield curve where data were downloaded from Haver Analytics.[6]

Our full estimation sample ranges from 1986:Q1 to 2012:Q2. Euro area GDP is taken as 400 times the quarterly change in the logarithm of GDP to obtain quarterly GDP growth at an annual rate. Note that euro area GDP values before 1995Q1 are taken from the euro area business cycle network backcasted GDP series. This allows us to have a long enough sample, which is helpful for the inference on regimes.

[6]The slope of the yield curve is defined as the difference between the yields on a 10-year government bond (national data are aggregated at the euro area level by Eurostat) and the yields on a 3-month money market interest rate (i.e., the EURIBOR).

4.2 In-sample estimates

First, we focus on the in-sample analysis. Table 4 reports the in-sample parameter estimates obtained with a MSMF-VAR(KF) model for the four indicators. The second regime is associated with higher volatility and lower GDP growth across all four indicators so that this regime can be interpreted as the recessionary regime.

Figure 1 shows the monthly estimates of GDP growth we obtain from the MSMF-VAR (KF) model with the ESI. These estimates are a byproduct of our estimation. We also report in Figure 1 the Eurocoin indicator, which is a monthly estimate of the medium-long run component of quarterly euro area GDP growth, see Altissimo et al. (2010). Our monthly estimates of GDP growth are relatively close to the Eurocoin indicator, albeit as expected they tend to be more volatile than Eurocoin.

Figure 2 shows the in-sample monthly probability of recession from the MSMF-VAR (KF) model with the ESI. The recession and expansion phases of the euro area economic activity are all very well captured by the MSMF-VAR (KF) model, including the latest euro area recession. However, note that this model identifies the 2001-03 period as recession, while the CEPR qualifies this episode as being a "prolonged pause in the growth of economic activity".[7]

In Table 5 we report the in-sample QPS and LPS for each model with MS parameters using the four indicators sequentially. We also report the QPS and LPS for an average (equal weights) of the smoothed probability over all models for a given indicator (see the last row of Panel A and Panel B)[8] and the LPS and QPS for an average (equal weights) of the smoothed probability over all indicators for a given model (see the last column of Panel A and Panel B). First, the ESI yields the best discrete estimates of the business cycle conditions pooling across all models (i.e., the lowest QPS and LPS, see the last row of Panel A and Panel B). Second, pooling across all indicators the MSMF-VAR (KF) obtains on average the lowest QPS, while the MSMF-VAR (SV-R) gets the lowest LPS (see the last column of Panel A and Panel B). Third, for a given indicator, taking a simple average of the smoothed probability over all models in general permits to substantially improve the classification of economic activity. Finally, and in line with the results for the US in Guérin and Marcellino (2013), the MS-MIDAS with the slope of the yield curve performs very well in terms of both LPS and QPS.

4.3 Forecasting results

The available sample is split between an estimation sample and an evaluation sample. The first estimation sample runs from 1986Q1 to 2006Q1 and it is recursively expanded

[7]One reason for differences in estimating turning points is that the CEPR business cycle dating committee concentrates on dating the classical business cycle (i.e., based on the level of economic activity), whereas regime-switching models estimate growth cycle where turning points are defined in deviations from trend.

[8]In this case, we exclude the probability from the standard univariate Markov-switching model (MS-AR) when taking the average.

until 2011Q2. Within the evaluation sample, we calculate three nowcasts as well as six forecasts, reflecting the information set available at the end of each month m of quarter t, $m = \{1, 2, 3\}$. For example, for the initial evaluation quarter 2006Q2, three nowcasts for quarter 2006Q2 are calculated corresponding to the information set available at the end of the months of April, May and June. Three forecasts for quarter 2006Q3 and three forecasts for quarter 2006Q4 are accordingly calculated using the information set available at the end of the months of April, May and June. As a result, the forecasting evaluation sample is $[T_1, T_2 + h]$ where T_1 is 2006Q2, T_2 is 2011Q3, and h denotes the maximum forecast horizon in quarters (i.e., 2).

Our forecasting exercise uses real-time data and takes into consideration the different publication lags for the monthly indicators we consider.[9] Actual GDP growth for evaluating our forecasts is taken from the September 2012 data vintage, with last observation T=2012:Q2. Forecasts are evaluated based on the RMSE, using an AR(1) model as the benchmark model.

Our forecasting exercise considers linear MIDAS models, Markov-Switching MIDAS models, linear MF-VAR models (also in stacked version), Markov-Switching MF-VAR models (both in state space and in stacked version). For the stacked version we have both restricted and unrestricted specifications. We also include as competitors a bivariate Markov-Switching model as well as an ADL(1,1) model and a VAR(1) model. Overall, we estimate for each indicator 14 models as well as an AR(1) model. We use the Clark and West (2007) test to compare the predictive accuracy for a given model against the more parsimonious (nested) AR(1) model.[10] Table 6 reports the out-of-sample forecasting results, in terms of RMSE.

For forecast horizons $h = \{0 - 5\}$, the ESI is the best indicator when combined with MSMF-VAR (SV-R) ($h = \{0, 1\}$), or MSMF-VAR (SV-U) ($h = \{2, 4, 5\}$) or MF-VAR (SV-R) ($h = \{3\}$). The good performance of the MSMF-VAR (SV) model is also in line with the Monte Carlo results.

Interestingly, for 2-quarter ahead forecasts ($h = 6 - 8$), M1 becomes the best indicator, in particular when combined with an ADL-MIDAS specification.

Overall, the results for nowcasting and short term forecasting GDP growth provide good support for the MSMF-VAR, with the MSMF-VAR (SV) best or second best for $h = \{0, 1, 2, 3, 4, 5\}$, in combination with the ESI. This is even more noticeable since there are a limited number of observations on regime 2 (recessions) in the sample, and as we have seen from the Monte Carlo experiments in this case simpler models, possibly even without MS, perform well.

To assess whether the MSMF-VAR also performs well at predicting turning points, Figures 3-8 report the recursively estimated vintages of the real-time probability of a euro

[9]Specifically, the economic sentiment indicator and the slope of the yield curve are readily available at the end of each month. We consider that industrial production and M1 monetary aggregate have a two-month publication lag (since we assume that forecasts are generated on the last day of each month).

[10]Admittedly, this test overlooks the real-time nature of the data. However, the Clark and McCracken (2009) test of equal predictive ability with real-time data is not straightforward to implement in the context of MIDAS models.

area recession obtained from the various MS specifications. The different vintages for the probabilities are obtained from the recursive forecasting exercise. Table 7 reports the average of the QPS and LPS calculated for each quarter of the evaluation sample. First, a simple univariate MS-AR model performs best in terms of QPS when compared to models that use the M1 monetary aggregate and industrial production as a monthly indicator. However, a visual inspection of Figure 4 shows that the model performs relatively poorly to detect the beginning of recessions. Second, when pooling the results across indicators, the MSMF-VAR (KF) model yields the best results in terms of QPS and LPS (see the last column of Table 7). In particular, Figure 3 shows that the MSMF-VAR (KF) model can indeed very well capture euro area recessionary episodes in a pseudo real-time forecasting context, including the ongoing euro area recession. Third, the stacked-system version of the MSMF-VAR is nearly always outperformed by its Kalman filter version counterpart (except when compared with the MSMF-VAR(SV-R) with the slope of the yield curve as a monthly indicator) in line with the simulation results. Fourth, as in Guérin and Marcellino (2013) for the US, the MS-MIDAS model with the slope of the yield curve also works very well for the euro area.

On balance, our results suggest that there are advantages in using both simple models based on GDP alone and more elaborated models involving different frequency mixes such as the MSMF-VAR(KF) model for detecting business cycle turning points. This is in line with the Monte Carlo results and the conclusions from Hamilton (2011).

5 Conclusions

In this paper we introduce regime switching parameters in the Mixed-Frequency VAR model. We first discuss estimation and inference for the resulting Markov-Switching Mixed Frequency VAR (MSMF-VAR) model, based either on a combination of the Kalman and Hamilton filters, as in Kim and Nelson (1999), or on the Hamilton filter applied on a stacked version of the MF-VAR.

Next, we assess the finite sample performance of the technique in Monte-Carlo experiments, relative to a large set of alternative models for mixed frequency data, with or without Markov Switching. We find that the estimation of the MSMF-VAR is fairly complex, such that its forecasting performance for the variables in levels is not so good unless there are frequent changes in regimes and a fairly long estimation sample. The MSMF-VAR (SV) seems to work better than the MSMF-VAR (KF), while the opposite is true when estimating regime changes. For the latter target, the MSMF-VAR (KF) becomes very competitive.

Finally, the MSMF-VAR model is applied to predict GDP growth and business cycle turning points in the euro area. Its performance is again compared with that of a number of competing models, including linear and regime switching mixed data sampling (MIDAS) models. In line with the Monte Carlo experiments, the MSMF-VAR (KF) is particularly useful to estimate the status of economic activity, while the MSMF-VAR (SV) seems to work well for nowcasting and short term forecasting the euro area GDP growth (when using the ESI as a timely monthly indicator).

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Appendix A: Kalman and Hamilton filters to estimate state-space models with regime switching

The equations of the basic Kalman filter can be found in a standard time series textbook such as Luetkepohl (2005). The general representation for a state-space model with regime switching in both measurement and transition equations is given by:

$$y_t = H(S_t)\beta_t + A(S_t)z_t + e_t$$
$$\beta_t = \tilde{\mu}(S_t) + F(S_t)\beta_{t-1} + G(S_t)v_t$$

where $e_t \sim N(0, R(S_t))$, $v_t \sim N(0, Q(S_t))$, and e_t and v_t are not correlated.

Kim and Nelson (1999) show how to combine the Kalman and Hamilton filters in a tractable way. The equations of the Kim and Nelson (1999) filtering procedure for state-space models with regime switching are:

$$\beta_{t|t-1}^{(i,j)} = \tilde{\mu}_j + F_j\beta_{t-1|t-1}^{i}$$

$$P_{t|t-1}^{i,j} = F_j P_{t-1|t-1}^{i} F_j' + GQ_jG_j'$$

$$\eta_{t|t-1}^{(i,j)} = y_t - H_j\beta_{t|t-1}^{(i,j)} - A_j z_t$$

$$f_{t|t-1}^{(i,j)} = H_j P_{t|t-1}^{(i,j)} H_j' + R_j$$

$$\beta_{t|t}^{(i,j)} = \beta_{t|t-1}^{(i,j)} + P_{t|t-1}^{(i,j)} H_j' [f_{t|t-1}^{(i,j)}]^{-1} \eta_{t|t-1}^{(i,j)}$$

$$P_{t|t}^{(i,j)} = (I - P_{t|t-1}^{(i,j)} H_j' [f_{t|t-1}^{(i,j)}]^{-1} H_j) P_{t|t-1}^{(i,j)}$$

When there is regime switching, it is also necessary to introduce approximations at the end of the Kalman and Hamilton filters to avoid the proliferation of cases to be considered:

$$\beta_{t|t}^{j} = \frac{\sum_{i=1}^{M} Pr[S_{t-1} = i, S_t = j|\Psi_t]\beta_{t|t}^{(i,j)}}{Pr[S_t = j|\Psi_t]}$$

$$P_{t|t}^{j} = \frac{\sum_{i=1}^{M} Pr[S_{t-1} = i, S_t = j|\Psi_t]\{P_{t|t}^{(i,j)} + (\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)})(\beta_{t|t}^{j} - \beta_{t|t}^{(i,j)})'\}}{Pr[S_t = j|\Psi_t]}$$

Appendix B: Additional details on the Monte Carlo experiments

In the Monte-Carlo experiments, we estimate 15 models to evaluate the forecasting performance of the Markov-Switching Mixed Frequency VAR models. Denote y_t the low frequency variable sampled at the low frequency unit, y_t^* the low frequency variable at the high frequency unit, x_t the high frequency variable aggregated at the low frequency unit via a simple arithmetic mean, and x_t^* the high frequency variable sampled at the high frequency unit. Also, assuming a mix of quarterly and monthly variables, denote $x_t^{*,1}$, $x_t^{*,2}$ and $x_t^{*,3}$, the releases of the high frequency variable in the first, second and third month of quarter t. When considering regime switches, we assume that S_t is an ergodic and irreducible Markov-chain with two regimes. The models under comparison are listed below.

- AR(1) model:
$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t, \epsilon_t \sim N(0, \sigma)$$
- ADL(1,1) model:
$$y_t = \alpha_0 + \alpha_1 y_{t-1} + \beta_1 x_t + \epsilon_t, \epsilon_t \sim N(0, \sigma)$$
- linear MIDAS model:
$$y_t = \beta_0 + \beta_1 B(L^{(1/m)}; \theta) x_{t-h}^* + \epsilon_t, \epsilon_t \sim N(0, \sigma)$$
- linear ADL-MIDAS model:
$$y_t = \beta_0 + \beta_1 B(L^{(1/m)}; \theta) x_{t-h}^* + \lambda y_{t-1} + \epsilon_t, \epsilon_t \sim N(0, \sigma)$$
- Univariate Markov-Switching model:
$$y_t = \alpha_0(S_t) + \alpha_1 y_{t-1} + \epsilon(S_t), \epsilon_t | S_t \sim N(0, \sigma(S_t))$$
- Markov-Switching MIDAS model
$$y_t = \beta_0(S_t) + \beta_1 B(L^{(1/m)}; \theta) x_{t-h}^* + \epsilon_t(S_t), \epsilon_t | S_t \sim N(0, \sigma(S_t))$$
- Markov-Switching ADL-MIDAS model
$$y_t = \beta_0(S_t) + \beta_1 B(L^{(1/m)}; \theta) x_{t-h}^* + \lambda y_{t-1} + \epsilon_t(S_t), \epsilon_t | S_t \sim N(0, \sigma(S_t))$$
- Kalman filter version of the linear Mixed Frequency VAR (MF-VAR (KF)):
$$Y_t^* = A_0 + A_1 Y_{t-1}^* + u_t, u_t \sim N(0, \Sigma)$$
where $Y_t^* = (y_t^*, x_t^*)'$
- Kalman filter version of the Markov-Switching Mixed Frequency VAR (MSMF-VAR (KF)):
$$Y_t^* = A_0(S_t) + A_1 Y_{t-1}^* + u_t(S_t), u_t | S_t \sim N(0, \Sigma(S_t))$$
where $Y_t^* = (y_t^*, x_t^*)'$

- VAR(1) model:
$$Y_t = A_0 + A_1 Y_{t-1} + u_t, u_t \sim N(0, \Sigma)$$
where $Y_t = (y_t, x_t)'$

- Markov-switching VAR(1) model:
$$Y_t = A_0(S_t) + A_1 Y_{t-1} + u_t(S_t), u_t|S_t \sim N(0, \Sigma(S_t))$$
where $Y_t = (y_t, x_t)'$

- Stacked vector system version of the linear Mixed Frequency VAR with no restrictions on the parameters (MF-VAR (SV-U)):
$$Z_t = A_0 + A_1 Z_{t-1} + u_t, u_t \sim N(0, \Sigma)$$
where $Z_t = (x_t^{*,1}, x_t^{*,2}, x_t^{*,3}, y_t)'$

- Stacked vector system version of the Markov-Switching Mixed Frequency VAR with no restrictions on the parameters (MSMF-VAR (SV-U)):
$$Z_t = A_0(S_t) + A_1 Z_{t-1} + u_t(S_t), u_t|S_t \sim N(0, \Sigma(S_t))$$
where $Z_t = (x_t^{*,1}, x_t^{*,2}, x_t^{*,3}, y_t)'$

- Stacked vector system version of the linear Mixed Frequency VAR with restrictions on some of the parameters (MF-VAR (SV-R)):
$$Z_t = A_0 + A_1 Z_{t-1} + u_t, u_t \sim N(0, \Sigma) \tag{24}$$
where $Z_t = (x_t^{*,1}, x_t^{*,2}, x_t^{*,3}, y_t)'$ and the A_1 matrix has a particular structure. We assume that the high frequency process is ARX(1) with the impact of the low frequency series constant throughout the period. Then, following Ghysels (2012), equation (24) can be rewritten as follows:
$$\begin{pmatrix} x_t^{*,1} \\ x_t^{*,2} \\ x_t^{*,3} \\ y_t \end{pmatrix} = \begin{pmatrix} 0 & 0 & \rho_H & a_1 \\ 0 & 0 & \rho_H^2 & a_2 \\ 0 & 0 & \rho_H^3 & a_3 \\ b_1 & b_2 & b_3 & \rho_L \end{pmatrix} * \begin{pmatrix} x_{t-1}^{*,1} \\ x_{t-1}^{*,2} \\ x_{t-1}^{*,3} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} u_t^{x^{*,1}} \\ u_t^{x^{*,2}} \\ u_t^{x^{*,3}} \\ u_t^{y} \end{pmatrix} \tag{25}$$
Likewise, in line with Ghysels (2012), we also impose the following restrictions on the variance-covariance matrix of the residuals:
$$\Sigma = \begin{pmatrix} \sigma_{xx} & \rho_H \sigma_{xx} & \rho_H^2 \sigma_{xx} & \sigma_{xy} \\ \rho_H \sigma_{xx} & (1+\rho_H)\sigma_{xx} & \rho_H \sigma_{xx} & \sigma_{xy} \\ \rho_H^2 \sigma_{xx} & \rho_H \sigma_{xx} & (1+\rho_H+\rho_H^2)\sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{xy} & \sigma_{xy} & \sigma_{yy} \end{pmatrix} \tag{26}$$

- Stacked vector system version of the Markov-Switching Mixed Frequency VAR with restrictions on some of the parameters (MSMF-VAR (SV-R)):

$$Z_t = A_0(S_t) + A_1 Z_{t-1} + u_t(S_t), u_t|S_t \sim N(0, \Sigma(S_t))$$

where $Z_t = (x_t^{*,1}, x_t^{*,2}, x_t^{*,3}, y_t)'$ and the A_1 matrix has a structure similar to that in equation (25) above.

The variance-covariance matrix of the residuals takes the following form:

$$\Sigma = \begin{pmatrix} \sigma_{xx}(S_t) & \rho_H \sigma_{xx}(S_t) & \rho_H^2 \sigma_{xx}(S_t) & \sigma_{xy}(S_t) \\ \rho_H \sigma_{xx}(S_t) & (1+\rho_H)\sigma_{xx}(S_t) & \rho_H \sigma_{xx}(S_t) & \sigma_{xy}(S_t) \\ \rho_H^2 \sigma_{xx}(S_t) & \rho_H \sigma_{xx}(S_t) & (1+\rho_H+\rho_H^2)\sigma_{xx}(S_t) & \sigma_{xy}(S_t) \\ \sigma_{xy}(S_t) & \sigma_{xy}(S_t) & \sigma_{xy}(S_t) & \sigma_{yy}(S_t) \end{pmatrix} \tag{27}$$

Appendix C: Nowcasting exercise with the stacked vector system version of the Mixed-frequency VAR model

In the case of the stacked vector system version of the mixed-frequency VAR model, nowcasts are obtained as follows.

First, denote the reduced-form version of the VAR(1) model as follows:

$$Z_t = AZ_{t-1} + e_t \tag{28}$$

Assume $E(e_t e_t') = V$ and $V = P^{-1}D(P^{-1})'$ where P has a unit main diagonal and D is a diagonal matrix. Premultiplying equation (28) by P, we obtain the structural VAR:

$$PZ_t = PAZ_{t-1} + \epsilon_t \tag{29}$$

The structural shocks, ϵ_t, are given by $\epsilon_t = Pe_t$

For simplicity of notation, we assume in the sequel that there is only one low-frequency variable and one high-frequency variable and we consider the standard macroeconomic forecasting context of a mix between quarterly and monthly variables but note that this framework can be easily extended to larger systems or different frequency mixes. We then assume that $Z_t = (x_t^{(1)}, x_t^{(2)}, x_t^{(3)}, y_t)'$, where:

- $x_t^{(1)}$ is the monthly indicator corresponding to the first month of the quarter.
- $x_t^{(2)}$ is the monthly indicator corresponding to the second month of the quarter.
- $x_t^{(3)}$ is the monthly indicator corresponding to the third month of the quarter.
- y_t is the quarterly indicator.

The first possibility is that we want to forecast y_{t+k} without knowing any observations for $x_{t+k}^{(1)}$, $x_{t+k}^{(2)}$ and $x_{t+k}^{(3)}$ where $k = \{1, 2, 3, ..., H\}$. This is the standard case where one uses the reduced-form VAR model described in equation (28).

By contrast, if one wants to forecast y_{t+k} provided that the first release of the high frequency indicator has been released (i.e. $x_{t+k}^{(1)}$), one can use certain elements of the structural VAR matrix P to capture the contemporaneous correlation between the variables for now-casting (and performing within quarter updates of the now-casts.

Let us define the P matrix of contemporaneous correlations that is obtained via a Choleski decomposition as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathsf{A} & 1 & 0 & 0 \\ \mathsf{B} & \mathsf{D} & 1 & 0 \\ \mathsf{C} & \mathsf{E} & \mathsf{F} & 1 \end{pmatrix}.$$

Then, equation (29) above can be rewritten as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathsf{A} & 1 & 0 & 0 \\ \mathsf{B} & \mathsf{D} & 1 & 0 \\ \mathsf{C} & \mathsf{E} & \mathsf{F} & 1 \end{pmatrix} * \begin{pmatrix} x_t^{(1)} \\ x_t^{(2)} \\ x_t^{(3)} \\ y_t \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{pmatrix} * \begin{pmatrix} x_{t-1}^{(1)} \\ x_{t-1}^{(2)} \\ x_{t-1}^{(3)} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} \epsilon_t^{S,x^{(1)}} \\ \epsilon_t^{S,x^{(2)}} \\ \epsilon_t^{S,x^{(3)}} \\ \epsilon_t^{S,y} \end{pmatrix} \tag{30}$$

where $PA = [\alpha]_{ij}$ is the matrix of structural-form VAR coefficients.

From the last equation in (30), the optimal forecast for y_{t+1} given $x_{t+1}^{(1)}$ is

$$\begin{aligned} y_{t+1} &= \alpha_{41}x_t^{(1)} + \alpha_{42}x_t^{(2)} + \alpha_{43}x_t^{(3)} + \alpha_{44}y_t - \mathsf{C}x_{t+1}^{(1)} + \\ &\quad -\mathsf{E}(E(x_{t+1}^{(2)}|x_{t+1}^{(1)}) - \mathsf{F}E(x_{t+1}^{(3)}|x_{t+1}^{(1)}), \end{aligned}$$

where

$$E(x_{t+1}^{(2)}|x_{t+1}^{(1)}) = \alpha_{21}x_t^{(1)} + \alpha_{22}x_t^{(2)} + \alpha_{23}x_t^{(3)} + \alpha_{24}y_t - \mathsf{A}x_{t+1}^{(1)}$$

and

$$E(x_{t+1}^{(3)}|x_{t+1}^{(1)}) = \alpha_{31}x_t^{(1)} + \alpha_{32}x_t^{(2)} + \alpha_{33}x_t^{(3)} + \alpha_{34}y_t - \mathsf{B}x_{t+1}^{(1)} - \mathbf{D}E(x_{t+1}^{(2)}|x_{t+1}^{(1)})$$

Along the same lines, the optimal nowcast for y_{t+1} given $x_{t+1}^{(1)}$ and $x_{t+1}^{(2)}$ is

$$\begin{aligned} y_{t+1} &= \alpha_{41}x_t^{(1)} + \alpha_{42}x_t^{(2)} + \alpha_{43}x_t^{(3)} + \alpha_{44}y_t - \mathsf{C}x_{t+1}^{(1)} + \\ &\quad -\mathsf{E}x_{t+1}^{(2)} - \mathsf{F}E(x_{t+1}^{(3)}|x_{t+1}^{(2)}), \end{aligned}$$

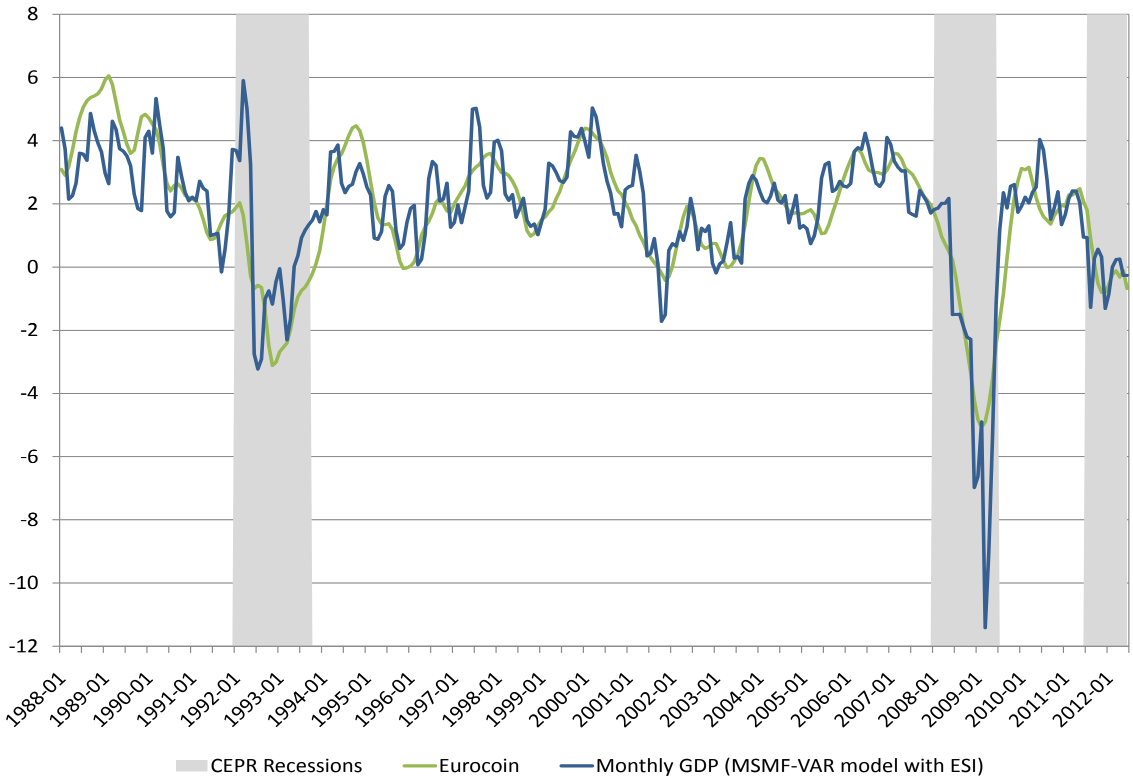
where

$$E(x_{t+1}^{(3)}|x_{t+1}^{(2)}) = \alpha_{31}x_t^{(1)} + \alpha_{32}x_t^{(2)} + \alpha_{33}x_t^{(3)} + \alpha_{34}GDP_t - \mathsf{B}x_{t+1}^{(1)} - \mathbf{D}x_{t+1}^{(2)}.$$

Finally, if we know $x_{t+1}^{(1)}$, $x_{t+1}^{(2)}$ and $x_{t+1}^{(3)}$ the nowcast for y_{t+1} is given by:

$$y_{t+1} = \alpha_{41}x_t^{(1)} + \alpha_{42}x_t^{(2)} + \alpha_{43}x_t^{(3)} + \alpha_{44}y_t - \mathsf{C}x_{t+1}^{(1)} - \mathsf{E}x_{t+1}^{(2)} - \mathsf{F}x_{t+1}^{(3)},$$

Figure 1: Monthly GDP from the Markov-switching Mixed-Frequency VAR



Note: This figure shows the monthly estimate of euro area quarterly GDP growth at an annual rate from the MSMF-VAR (KF) model with quarterly GDP and monthly ESI along with the Eurocoin indicator also at an annual rate.

Table 1: In-sample Results - Monte Carlo experiments

		AR(1)	ADL	MS-AR	MIDAS	MS-MIDAS	ADL-MIDAS	MSADL-MIDAS	VAR	MS-VAR	MF-VAR (KF)	MSMF-VAR (KF)	MF-VAR (SV-U)	MSMF-VAR (SV-U)	MF-VAR (SV-R)	MSMF-VAR (SV-R)
Panel A: MS DGP with $p_{11} = 0.95$ and $p_{22} = 0.95$																
$\alpha = 0.1$	IS-MSE	5.966	5.648	2.743	6.094	2.553	5.773	2.623	5.648	3.283	-	-	5.792	3.297	5.832	**2.357**
	QPS	-	-	.236	-	.237	-	.224	-	.240	-	**.205**	-	.224	-	.315
	LPS	-	-	.624	-	.588	-	.526	-	.755	-	**.331**	-	.811	-	.879
$\alpha = 0.5$	IS-MSE	10.959	10.801	4.747	10.652	4.130	9.832	3.962	10.801	4.501	-	-	9.612	4.786	9.774	**3.325**
	QPS	-	-	.303	-	.247	-	.232	-	.254	-	**.221**	-	.234	-	.341
	LPS	-	-	.906	-	.654	-	.588	-	.757	-	**.353**	-	.854	-	.972
Panel B: MS DGP with $p_{11} = 0.95$ and $p_{22} = 0.85$																
$\alpha = 0.1$	IS-MSE	6.081	5.881	2.642	5.929	2.408	5.615	2.563	5.828	3.045	-	-	5.746	3.218	5.766	**2.249**
	QPS	-	-	.233	-	.244	-	.228	-	.228	-	**.183**	-	.224	-	.326
	LPS	-	-	.575	-	.567	-	.516	-	.707	-	**.302**	-	.835	-	.823
$\alpha = 0.5$	IS-MSE	10.876	10.508	4.672	10.650	4.255	9.943	4.134	10.688	4.555	-	-	9.662	4.830	9.822	**3.215**
	QPS	-	-	.309	-	.247	-	.228	-	.237	-	**.191**	-	.234	-	.348
	LPS	-	-	.765	-	.672	-	.597	-	.792	-	**.319**	-	.871	-	.980
Panel C: Mixed-frequency DGP with $p_{11} = 0.95$ and $p_{22} = 0.95$																
	IS-MSE	1.666	1.597	**.692**	2.683	.904	1.592	.737	1.601	.818	-	-	1.571	.860	1.573	.838
	QPS	-	-	.306	-	.464	-	.434	-	**.239**	-	.739	-	.256	-	.243
	LPS	-	-	.630	-	1.194	-	.697	-	**.567**	-	1.151	-	.640	-	.568
Panel D: Mixed-frequency DGP with $p_{11} = 0.95$ and $p_{22} = 0.85$																
	IS-MSE	1.661	1.595	**.698**	2.679	.902	1.582	.772	1.600	.806	-	-	1.562	.857	1.564	.834
	QPS	-	-	.319	-	.460	-	.474	-	**.244**	-	.891	-	.255	-	.246
	LPS	-	-	.668	-	1.194	-	.704	-	**.570**	-	1.581	-	.639	-	.573

Note: This table reports the in-sample MSE (IS-MSE) as defined in equation (17). For the mixed-frequency VAR models estimated via the Kalman filter, we do not report the in-sample MSE since the estimated low frequency variable is equal to the actual low frequency variable by construction. For regime switching models, we also report QPS and LPS as defined in equations (18) and (19). Bold entries are the smallest IS-MSE, QPS and LPS for the indicated DGP.

Table 2: Out-of-sample Results - Monte Carlo experiments

		AR(1)	ADL	MS-AR	MIDAS	MS-MIDAS	ADL-MIDAS	MSADL-MIDAS	Av. MIDAS	VAR	MS-VAR	Av. VAR	MF-VAR (KF)	MSMF-VAR (KF)	Av. MF-VAR (KF)	MF-VAR (SV-U)	MSMF-VAR (SV-U)	MF-VAR (SV-R)	MSMF-VAR (SV-R)	Av. MF-VAR (SV)
Panel A: MS DGP with $p_{11} = 0.95$ and $p_{22} = 0.95$																				
$\alpha = 0.1$	MSFE	6.047	5.829	6.300	6.842	5.951	6.049	5.889	5.997	5.858	5.590	5.646	7.738	6.332	6.636	5.882	**5.256**	5.607	5.600	5.428
	QPS	-	-	.235	-	.231	-	.229	.233	-	.228	.228	-	**.200**	.200	-	.220	-	.314	.235
	LPS	-	-	.583	-	.610	-	.545	.503	-	.699	.699	-	**.336**	.336	-	.849	-	.952	.526
$\alpha = 0.5$	MSFE	10.862	10.699	10.300	11.555	10.316	10.053	9.759	10.215	10.603	10.081	10.146	19.778	20.306	19.560	9.595	**8.722**	9.360	9.632	8.892
	QPS	-	-	.318	-	.251	-	.234	.240	-	.235	.235	-	**.205**	.205	-	.234	-	.356	.247
	LPS	-	-	.980	-	.705	-	.630	.558	-	.691	.691	-	**.342**	.342	-	.843	-	1.090	.556
Panel B: MS DGP with $p_{11} = 0.95$ and $p_{22} = 0.85$																				
$\alpha = 0.1$	MSFE	6.050	5.783	5.688	6.608	5.796	5.958	5.798	5.829	5.741	**5.547**	5.554	7.902	6.425	6.754	6.245	5.576	5.956	6.019	5.863
	QPS	-	-	.241	-	.244	-	.229	.234	-	.228	.228	-	**.203**	.203	-	.232	-	.326	.237
	LPS	-	-	.585	-	.593	-	.543	.507	-	.680	.680	-	**.339**	.339	-	.853	-	.919	.543
$\alpha = 0.5$	MSFE	10.760	10.598	10.206	11.788	10.591	10.180	9.849	10.409	10.597	10.026	10.161	20.081	20.414	19.873	10.383	**9.617**	10.426	10.627	10.010
	QPS	-	-	.332	-	.259	-	.238	.245	-	.234	.234	-	**.203**	.203	-	.233	-	.345	.240
	LPS	-	-	1.091	-	.725	-	.638	.585	-	.713	.713	-	**.341**	.341	-	.851	-	.950	.555
Panel C: Mixed-frequency DGP with $p_{11} = 0.95$ and $p_{22} = 0.95$																				
	MSFE	1.658	1.597	1.637	2.770	2.009	1.605	1.627	1.770	1.635	**1.595**	1.600	1.943	3.600	2.451	1.668	1.624	1.637	1.600	1.603
	QPS	-	-	.355	-	.471	-	.493	.487	-	**.246**	.246	-	.481	-	-	.269	-	.247	.251
	LPS	-	-	.730	-	1.240	-	.713	.702	-	.588	.588	-	.699	-	-	.693	-	.587	**.563**
Panel D: Mixed-frequency DGP with $p_{11} = 0.95$ and $p_{22} = 0.85$																				
	MSFE	1.655	1.595	1.634	2.850	2.024	1.621	1.627	1.783	1.635	1.608	1.605	1.932	3.539	2.433	1.654	1.590	1.616	**1.579**	1.581
	QPS	-	-	.350	-	.483	-	.487	.481	-	.246	.246	-	.483	.483	-	.265	-	**.246**	.249
	LPS	-	-	.730	-	1.262	-	.714	.702	-	.583	.583	-	.705	.705	-	.705	-	**.582**	.566

Note: This table reports the out-of-sample MSE. For regime switching models, we report QPS and LPS as defined in equations (22) and (23). Bold entries are the smallest MSE, QPS and LPS for the indicated DGP. Appendix B describes each model under scrutiny. Av. MIDAS, Av. MF-VAR(KF) and Av. MF-VAR(SV) reports the MSE, QPS and LPS obtained by taking averages (equal weights) of the continuous forecasts and smoothed probabilities over MIDAS, MF-VAR(KF) and MF-VAR(SV) models, respectively.

Table 3: List of monthly indicators

Indicator	Transformation
Economic Sentiment Indicator	quarterly change in the logarithm of monthly ESI
$M1$ Monetary Aggregate	quarterly change in the logarithm of monthly $M1$
Industrial Production	quarterly change in the logarithm of monthly IP
Slope of the yield curve	quarterly change in the level of the slope of the yield curve

This table reports the list of monthly indicators we use in the empirical application. The second column of the table shows the transformation we applied to each indicator. Data were downloaded from the ECB real-time database and Haver Analytics.

Table 4: In-sample results - MSMF-VAR (KF) model

	ESI	$M1$ Monetary Aggregate	Industrial Production	Slope of the yield curve
p_{11}	.975	.961	.992	.970
	[.012]	[.016]	[.006]	[.015]
p_{22}	.922	.724	.904	.952
	[.039]	[.108]	[.078]	[.029]
μ_y^1	2.597	2.096	2.053	2.234
	[.239]	[.280]	[.242]	[.227]
μ_x^1	-.458	1.628	.465	-0.067
	[.800]	[.141]	[.171]	[.069]
μ_y^2	-.999	.545	-.486	1.094
	[.706]	[1.383]	[2.463]	[.732]
μ_x^2	-1.868	2.880	-1.078	.131
	[1.227]	[.373]	[1.296]	[.145]
σ_y^1	.891	.962	1.819	.540
	[.168]	[.274]	[.167]	[.085]
σ_x^1	1.131	.610	.693	.211
	[.057]	[.033]	[.046]	[0.013]
σ_y^2	1.994	2.583	4.603	1.598
	[.435]	[.820]	[1.024]	[0.275]
σ_x^2	2.993	1.854	2.244	.401
	[.273]	[.299]	[.597]	[.033]

This table reports the in-sample parameter estimates for the MSMF-VAR (KF) model with quarterly GDP and monthly economic sentiment indicator. p_{11} is the transition probability of staying in the first regime. p_{22} is the transition probability of staying in the second regime. The μ's are the intercepts for each equation of the VAR in each regime, while the σ's are the estimates for the variance of the innovation in each regime. Standard deviations are reported in brackets and are calculated from the outer product estimate of the Hessian.

Table 5: In-sample QPS and LPS

	ESI	$M1$ Monetary Aggregate	Industrial Production	Slope of the yield curve	Pooling across indicators
Panel A: Quadratic Probability Score (QPS)					
MS-AR	.239	.239	.239	.239	.239
MSADL-MIDAS	.289	.260	**.223**	.233	.245
MS-MIDAS	.688	.457	.446	**.159**	.342
MS-VAR	.489	.558	.354	.268	.227
MSMF-VAR (KF)	.201	.364	.458	.390	.199
MSMF-VAR (SV-U)	.273	.415	.637	.663	.255
MSMF-VAR (SV-R)	.262	.494	.447	.305	.203
Pooling across models	**.172**	**.210**	.341	.233	**.191**
Panel B: Log Probability Score (LPS)					
MS-AR	.601	.601	.601	.601	.601
MSADL-MIDAS	.676	.686	.533	.554	.567
MS-MIDAS	1.673	1.101	1.033	**.297**	.581
MS-VAR	1.428	1.114	.867	.612	.363
MSMF-VAR(KF)	.439	.784	.852	.632	.321
MSMF-VAR (SV-U)	.699	1.449	1.845	2.452	.397
MSMF-VAR (SV-R)	.596	1.800	1.265	.684	**.316**
Pooling across models	**.304**	**.347**	**.516**	.404	.324

This table reports the in-sample quadratic probability score (QPS) and log probability score (LPS). We use the chronology of the euro area business cycle from the CEPR business cycle dating committee to identify the euro area expansions and recessions. This chronology is only available at the quarterly frequency. To calculate QPS and LPS for MSMF-VAR (KF) models that directly estimate the model at the monthly frequency, we disaggregated the CEPR business cycle chronology at the monthly frequency by assuming that the euro area was in recession in month m belonging to quarter t if the CEPR business cycle dating committee considered that the euro area was in recession in quarter t. The probabilities of a euro area recession are from the September 2012 data vintage with last observation June 2012 for the MSMF-VAR (KF) model and 2012Q2 for the other models. Bold entries are the smallest QPS and LPS for the indicated indicator.

Table 6: Forecasting euro area GDP growth - RMSE

Forecasting horizon (months)	0	1	2	3	4	5	6	7	8
				Economic Sentiment Indicator					
MIDAS	.890***	.804***	.805***	.650**	.774**	.820***	.873***	.893***	.902***
ADL-MIDAS	.724***	.664***	.641**	.557**	.725**	.811***	.836**	.887***	.893**
MS-MIDAS	.755**	.720**	.669**	.772**	.824**	.810**	.902**	.979**	.953**
MSADL-MIDAS	.655**	.623**	.545*	.595**	.698**	.817**	.872**	.912**	.954**
VAR	.796**	.796**	.796**	.996	.996	.996	1.034	1.034	1.034
MS-VAR	.773**	.773**	.773**	1.056	1.056	1.056	1.178	1.178	1.178
MF-VAR (KF)	.651**	.694**	.836**	.853**	.955**	1.014	.952	.963**	.958*
MSMF-VAR (KF)	.673**	.709**	.894*	.666*	.942	1.018	1.084	1.085	1.114
MF-VAR (SV-U)	.556**	.545**	.627**	.630**	.720**	.846**	.868***	.887***	.974**
MSMF-VAR (SV-U)	.643**	.542**	.533***	.562**	.592**	.704**	.844*	.842**	.908**
MF-VAR (SV-R)	.429**	.513**	.626***	.528**	.739**	.866**	.898***	1.044**	1.081**
MSMF-VAR (SV-R)	.413**	.433**	.539**	.679**	.737**	.981**	.908**	.930**	.998**
ADL(1,1)	.781**	.781**	.781**	.923**	.923**	.923**	.973*	.973*	.973*
				Industrial Production					
MIDAS	.552*	.733	1.094	1.117	1.254	1.215	1.082	1.072	1.035
ADL-MIDAS	.506*	.733	1.061	1.125	1.231	1.203	1.106	1.082	1.025
MS-MIDAS	.586*	.977	1.200	.829	.929	.991	1.044	1.134	1.184
MSADL-MIDAS	.590*	1.052	1.147	.911	.917	.952	1.023	1.157	1.148
VAR	1.121	1.121	1.121	1.183	1.183	1.183	1.205	1.205	1.205
MS-VAR	.846	.846	.846	1.010	1.010	1.010	1.277	1.277	1.277
MF-VAR (KF)	.662	.741	.733	.791**	.831**	.879**	.826**	.832**	.831**
MSMF-VAR (KF)	1.093	1.251	1.362	.706*	.867**	.890**	.900**	.878**	.885**
MF-VAR (SV-U)	.729*	1.116	1.253	1.205	1.345	1.387	1.296	1.343	1.377
MSMF-VAR (SV-U)	.853*	1.357	1.474	1.233	1.383	1.416	1.538	1.628	1.701
MF-VAR (SV-R)	.703*	1.097	1.181	1.242	1.347	1.348	1.305	1.316	1.325
MSMF-VAR (SV-R)	.678*	.924	1.006	1.132	1.266	1.264	1.224	1.384	1.391
ADL(1,1)	1.029	1.029	1.029	1.088	1.088	1.088	.976*	.976*	.976*
				Slope of the yield curve					
MIDAS	1.156	1.144	1.317	1.046	1.042	1.033	1.050	1.048*	1.050
ADL-MIDAS	.922**	1.020**	1.026	1.051	1.007*	1.000*	.992	.994	.986
MS-MIDAS	.994	.800	1.493	1.004	1.011	0.874	1.163	1.185	1.179
MSADL-MIDAS	.749	.552	1.077	1.029	1.007	1.059*	.980	1.094	1.066
VAR	1.011**	1.011**	1.011**	1.047*	1.047*	1.047*	1.025	1.025	1.025
MS-VAR	.922**	.922**	.922**	.988	.988	.988	1.018	1.018	1.018
MF-VAR (KF)	1.080	1.106	1.078	.944*	.946	.955	.931**	.932**	.932**
MSMF-VAR (KF)	1.223	1.384	1.435	1.045	1.032	1.025	.994	.990	.992
MF-VAR (SV-U)	1.258	1.233	1.282	1.061	1.070	1.142	1.029	1.028	1.065
MSMF-VAR (SV-U)	1.173	1.136	1.268	1.048	1.046	1.060	1.073	1.074	1.112
MF-VAR (SV-R)	1.271	1.256	1.250	1.071	1.080	1.189	1.031	1.043	1.108
MSMF-VAR (SV-R)	1.162	1.156	1.130	1.028	1.036	1.005	1.030	1.044	1.062
ADL(1,1)	1.011	1.011	1.011	.978	.978	.978	.972*	.972*	.972*
Univariate MS model	.851*	.851*	.851*	.947	.947	.947	1.042	1.042	1.042

This table reports the relative mean squared forecast error (RMSE) for forecasting euro area GDP growth. The benchmark model is an AR(1) model. Bold entries are the smallest RMSE for the indicated horizon and indicator. Asterisks *, **, *** indicate cases in which the Clark and West (2007) test rejected the null hypothesis of equal forecast accuracy at the 10%, 5% and 1% significance levels, respectively (one-sided tests, the parsimonious model is always the AR(1) model).

Table 6: Forecasting euro area GDP growth - RMSE (continued)

Forecasting horizon (months)	0	1	2	3	4	5	6	7	8
				Monetary Aggregate M1					
MIDAS	.960*	.980*	.954*	.793**	.766**	.736**	.702**	.698**	.717**
ADL-MIDAS	.772***	.782***	.875**	.771**	.733***	.713***	.692***	.675***	.702***
MS-MIDAS	.839**	.859***	.895***	.948**	.946**	.822***	.831***	.859***	.758***
MSADL-MIDAS	.633*	.871	.621**	1.007	.734***	.756***	.752***	.767***	.740***
VAR	.920**	.920**	.920**	.946	.946	.946	1.342	1.342	1.342
MS-VAR	1.111	1.111	1.111	.953*	.953*	.953*	1.007	1.007	1.007
MF-VAR (KF)	.787**	.722**	.866***	.704**	.714***	.831***	.854**	.938**	1.030*
MSMF-VAR (KF)	1.273	1.266	1.226	1.037	1.039	.968	.951*	.959*	.953*
MF-VAR (SV-U)	1.129	.997	.989	.990	1.129	1.106	1.012	.931**	.933**
MSMF-VAR (SV-U)	.963*	.933*	.963	.937*	.948*	.940*	1.007	.962*	.969*
MF-VAR (SV-R)	1.030	.957	.940	.875**	.891**	.889**	.878***	.765**	.789**
MSMF-VAR (SV-R)	.895**	.856***	.879**	.783**	.848**	.927**	.726**	.706**	.922**
ADL(1,1)	1.011	1.011	1.011	.819***	.819***	.819***	.847***	.847***	.847***
				Model average across all indicators (equal weights)					
MIDAS	.732***	.783***	.828***	.721**	.801***	.828***	.831***	.849***	.870***
ADL-MIDAS	.627**	.698**	.782**	.748**	.801**	.839**	.842**	.848**	.855***
MS-MIDAS	.530**	.557**	.725**	.722**	.803**	.799**	.863***	.909***	.893***
MSADL-MIDAS	.471**	.550*	.636*	.896	.793**	.841**	.835**	.867**	.873***
VAR	.862**	.862**	.862**	.955	.955	.955	1.059	1.059	1.059
MS-VAR	.838**	.838**	.838**	.956	.956	.956	1.076	1.076	1.076
MF-VAR (KF)	.537**	.644**	.734**	.698**	.800*	.865**	.838**	.844**	.855**
MSMF-VAR (KF)	.953	.955	1.063	.966	.969**	.979	.984	.999*	1.007
MF-VAR (SV-U)	.660**	.683**	.706**	.781**	.871**	.920**	.959**	.946**	.954*
MSMF-VAR (SV-U)	.616**	.658**	.704**	.798**	.846**	.850**	1.023	1.015	1.013
MF-VAR (SV-R)	.623**	.659**	.683**	.788**	.858**	.889*	.926**	.907**	.931*
MSMF-VAR (SV-R)	.547**	.586**	.613**	.769**	.852**	.858*	.912**	.936*	.965*
ADL(1,1)	.900*	.900*	.900*	.914**	.914**	.914**	.927***	.927***	.927***
Univariate MS model	.851*	.851*	.851*	.947	.947	.947	1.042	1.042	1.042

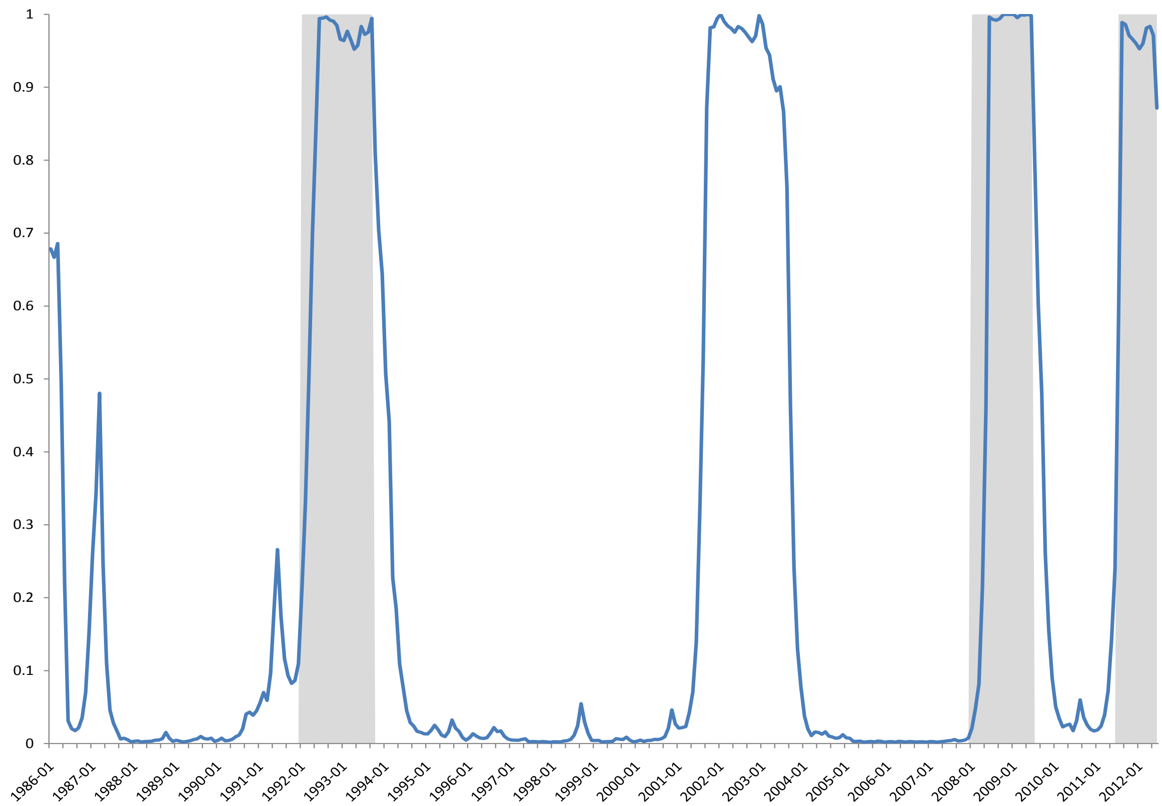
This table reports the relative mean squared forecast error (RMSE) for forecasting euro area GDP growth. The benchmark model is an AR(1) model. Bold entries are the smallest RMSE for the indicated horizon and indicator. Asterisks *, **, *** indicate cases in which the Clark and West (2007) test rejected the null hypothesis of equal forecast accuracy at the 10%, 5% and 1% significance levels, respectively (one-sided tests, the parsimonious model is always the AR(1) model).

Table 7: QPS and LPS - Recursive forecasting exercise

	ESI	$M1$ Monetary Aggregate	Industrial Production	Slope of the yield curve	Pooling across indicators
Panel A: Quadratic Probability Score (QPS)					
MS-AR	.201	**.201**	**.201**	.201	.201
MSADL-MIDAS	.394	.254	.251	.205	.237
MS-MIDAS	.615	.444	.486	**.194**	.353
MS-VAR	.480	.624	.371	.422	.252
MSMF-VAR (KF)	.236	.330	.379	.597	**.187**
MSMF-VAR (SV-U)	.364	.444	.670	.895	.299
MSMF-VAR (SV-R)	.276	.511	.454	.386	.217
Pooling across models	**.194**	.211	.285	.294	.193
Panel B: Log Probability Score (LPS)					
MS-AR	.597	.597	.597	.597	.597
MSADL-MIDAS	1.018	.774	.668	.597	.636
MS-MIDAS	1.725	1.296	1.269	**.451**	.671
MS-VAR	1.480	1.272	.964	1.045	.401
MSMF-VAR(KF)	.526	.769	.879	1.282	**.313**
MSMF-VAR (SV-U)	1.128	1.558	2.330	3.287	.488
MSMF-VAR (SV-R)	.812	1.929	1.276	1.000	.336
Pooling across models	**.322**	**.348**	**.429**	.461	.332

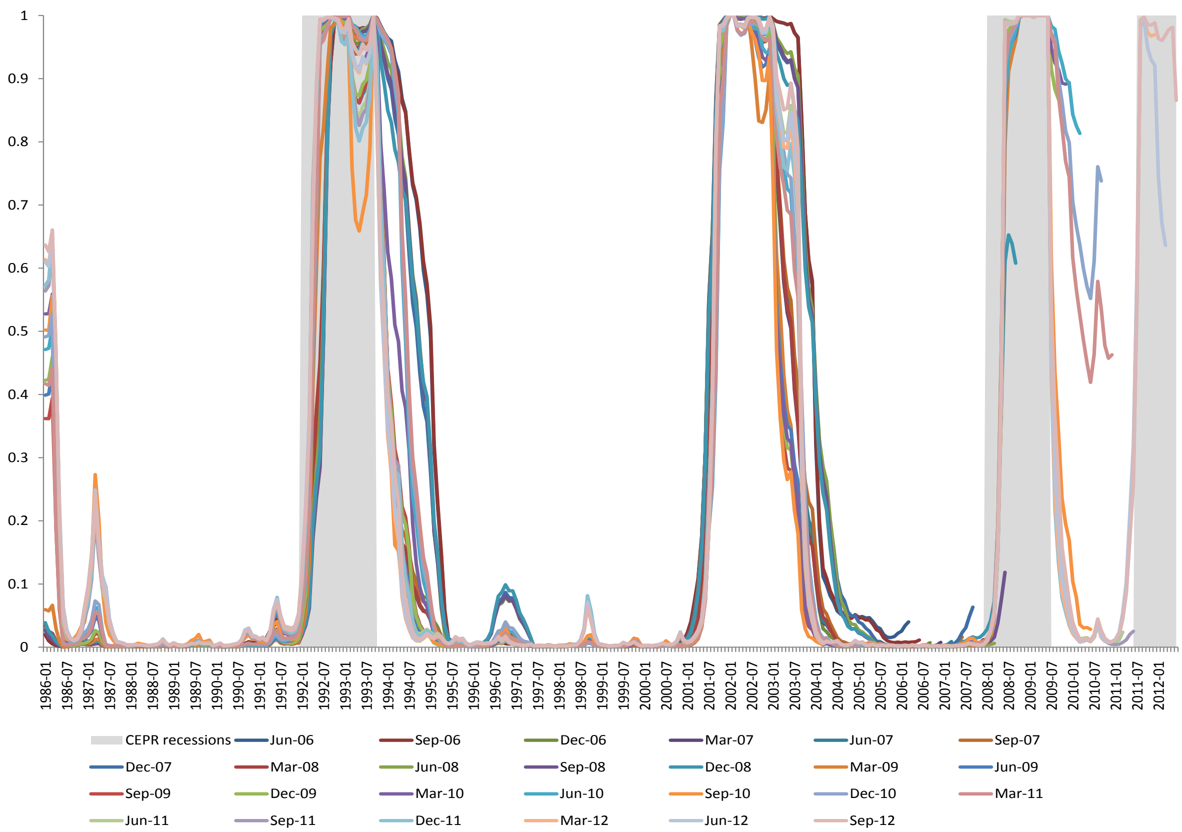
This table reports the quadratic probability score (QPS) and log probability score (LPS) calculated for each quarter of the recursive forecasting exercise, then averaged over the size of the evaluation sample. We use the chronology of the euro area business cycle from the CEPR business cycle dating committee to identify the euro area expansions and recessions. This chronology is only available at the quarterly frequency. To calculate QPS and LPS for MSMF-VAR (KF) models that directly estimate the model at the monthly frequency, we disaggregated the CEPR business cycle chronology at the monthly frequency by assuming that the euro area was in recession in month m belonging to quarter t if the CEPR business cycle dating committee considered that the euro area was in recession in quarter t. Bold entries are the smallest QPS and LPS for the indicated indicator.

Figure 2: Monthly probability of recession - in sample estimates



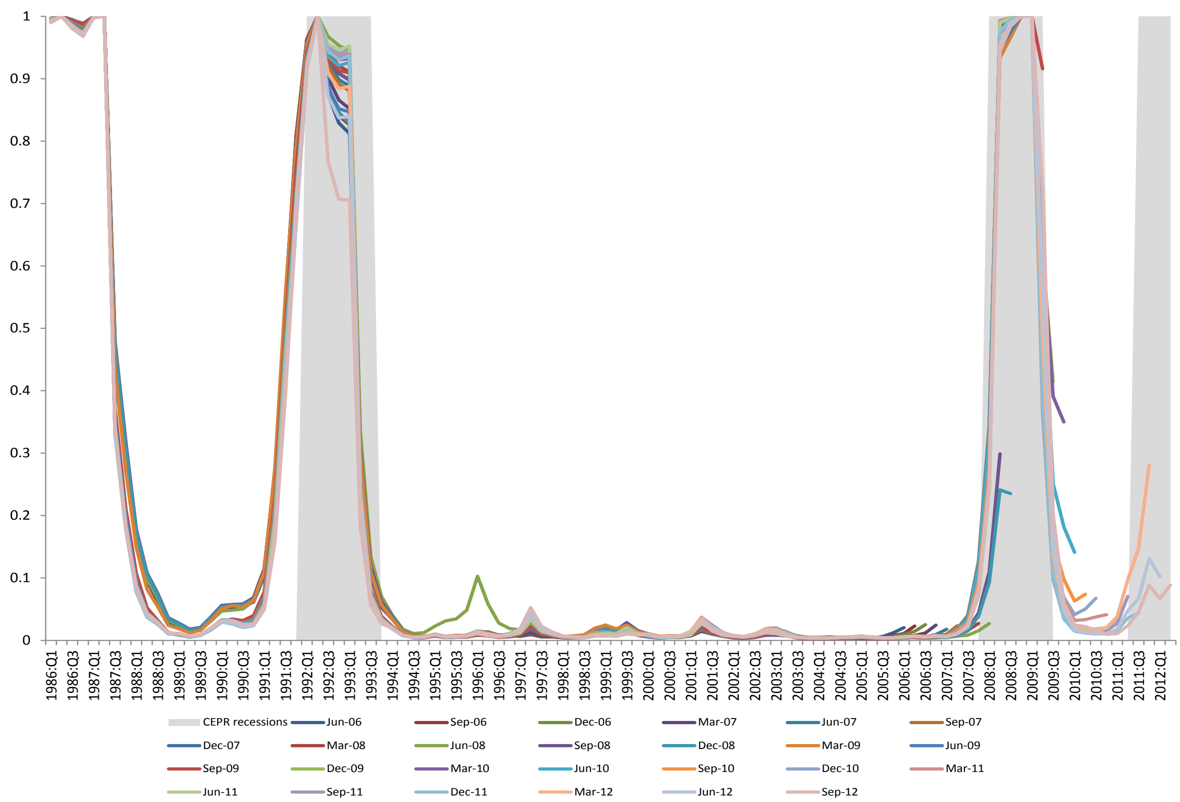
Note: The above probability of a euro area recession are from a MSMF-VAR model with quarterly GDP and monthly ESI. The shaded areas represent the CEPR recessions. Note that the model identifies the 2001-2003 period as a recession while the CEPR qualifies this episode as being a "prolonged pause in the growth of economic activity".

Figure 3: Euro area probability of recession obtained from the recursive forecasting exercise - MSMF-VAR (KF) model



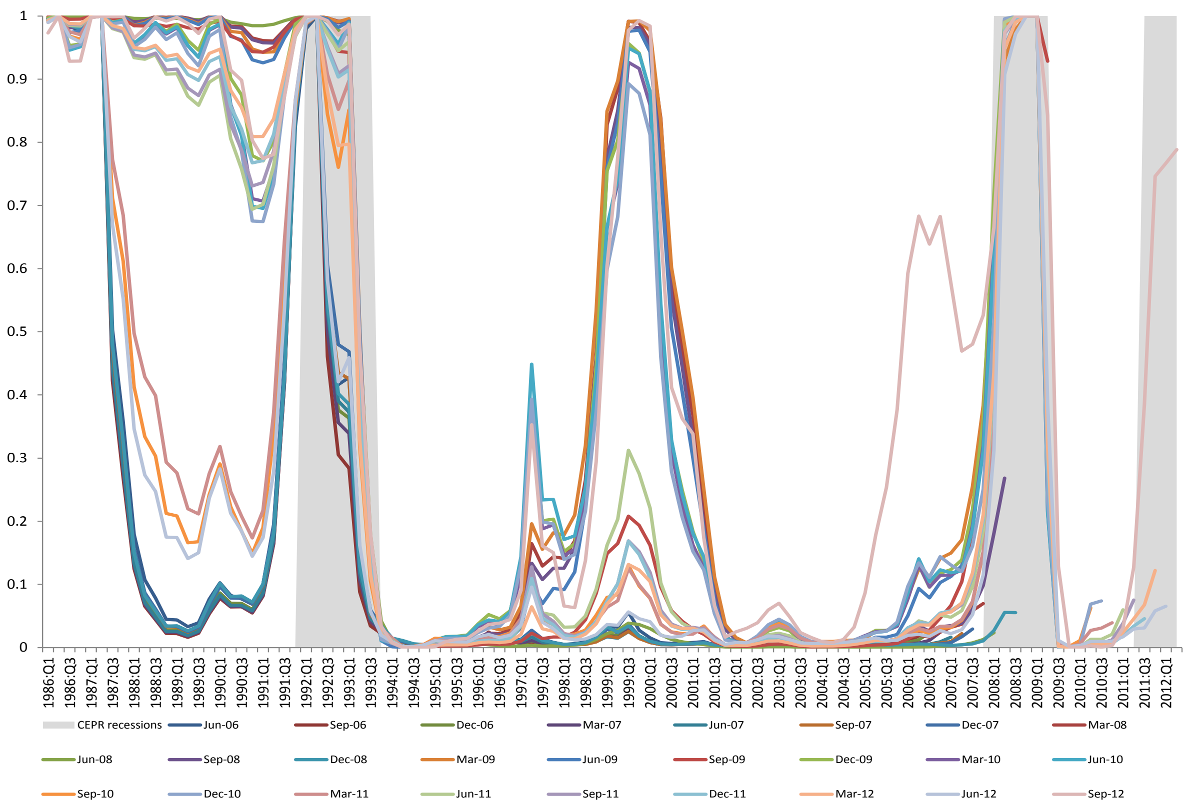
Note: This figure reports the monthly real-time probability of a euro area recession obtained from the MSMF-VAR (KF) model with quarterly GDP and monthly ESI. The probabilities are from the recursive forecasting exercise using real-time data vintages. For example, the June 2006 data vintage uses information up to 2006:Q1 for GDP and March 2006 for ESI.

Figure 4: Euro area probability of recession obtained from the recursive forecasting exercise - Univariate MS model



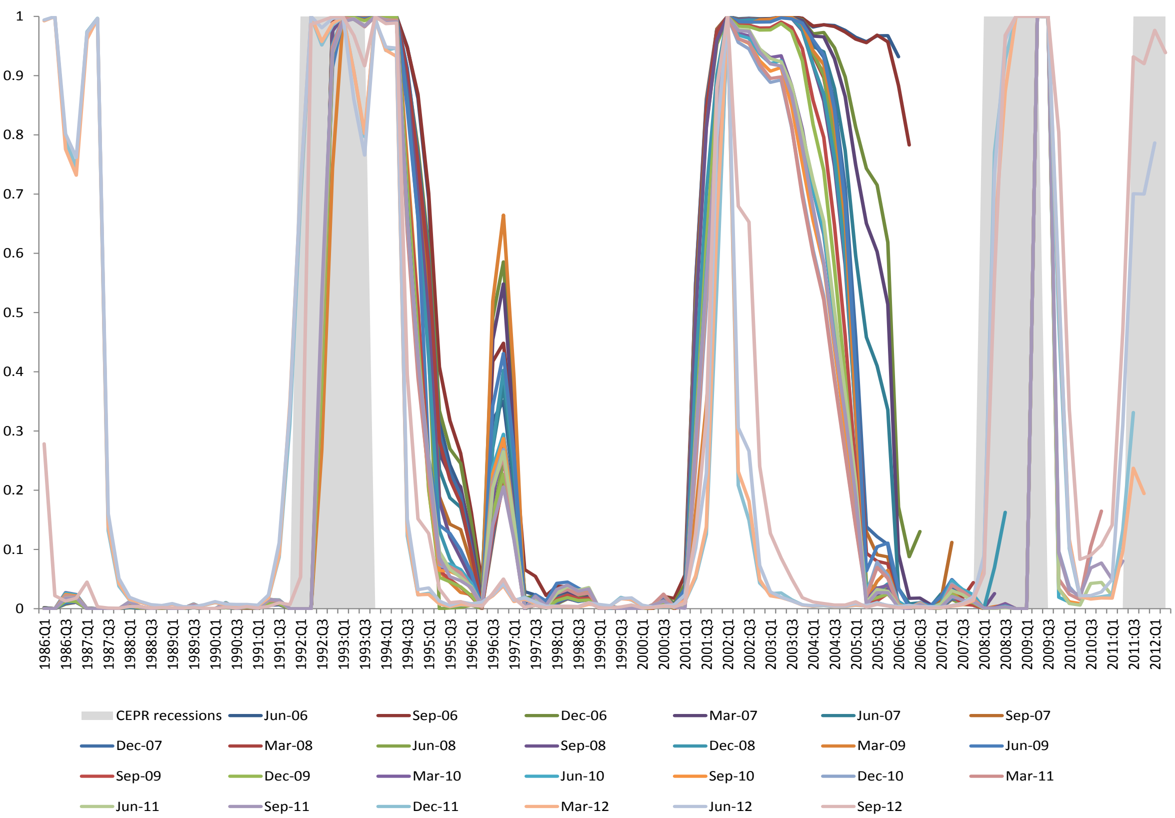
Note: This figure reports the quarterly real-time probability of a euro area recession obtained from a univariate MS model. The probabilities are from the recursive forecasting exercise using real-time data vintages. For example, the June 2006 data vintage uses information up to 2006:Q1 for GDP.

Figure 5: Euro area probability of recession obtained from the recursive forecasting exercise - MS-MIDAS model



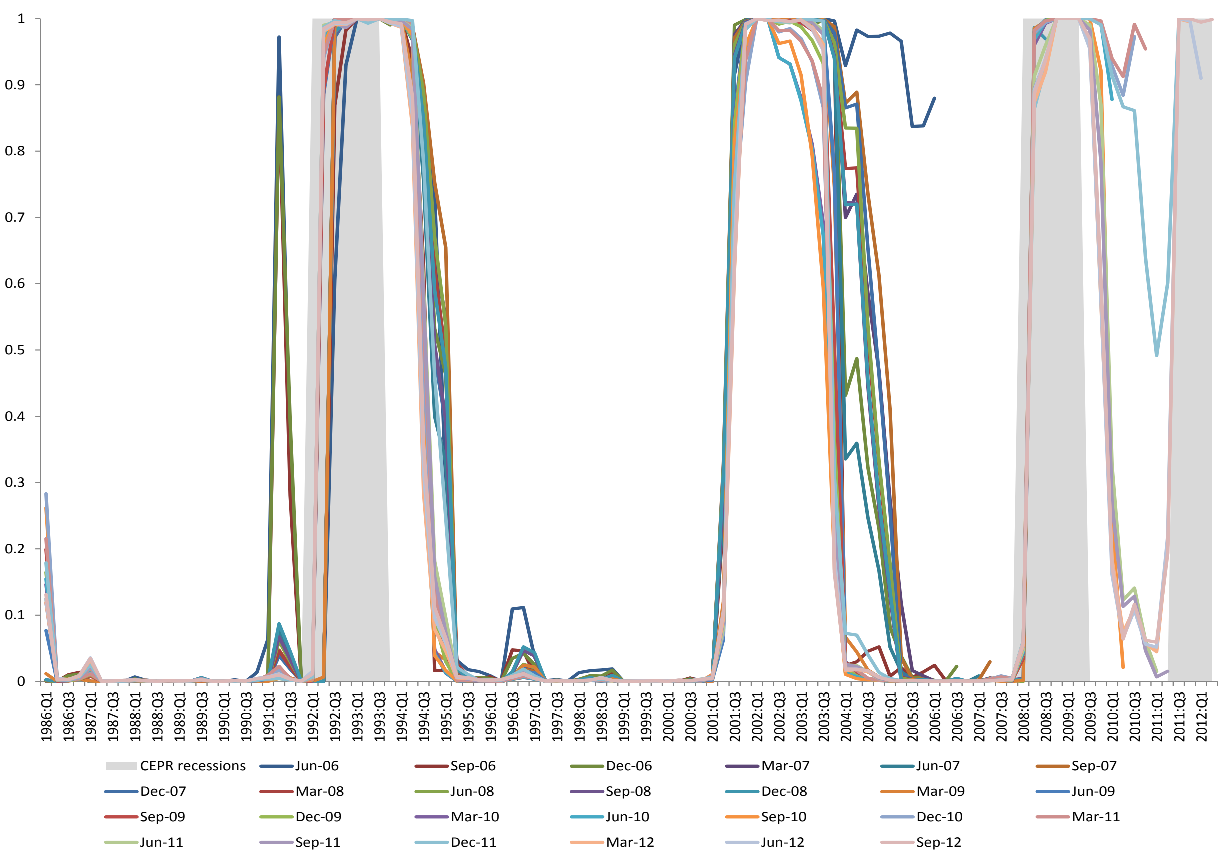
Note: This figure reports the quarterly real-time probability of a euro area recession obtained from a MS-MIDAS model with quarterly GDP and monthly ESI. The probabilities are from the recursive forecasting exercise using real-time data vintages. For example, the June 2006 data vintage uses information up to 2006:Q1 for GDP and March 2006 for ESI.

Figure 6: Euro area probability of recession obtained from the recursive forecasting exercise - Bivariate MS model



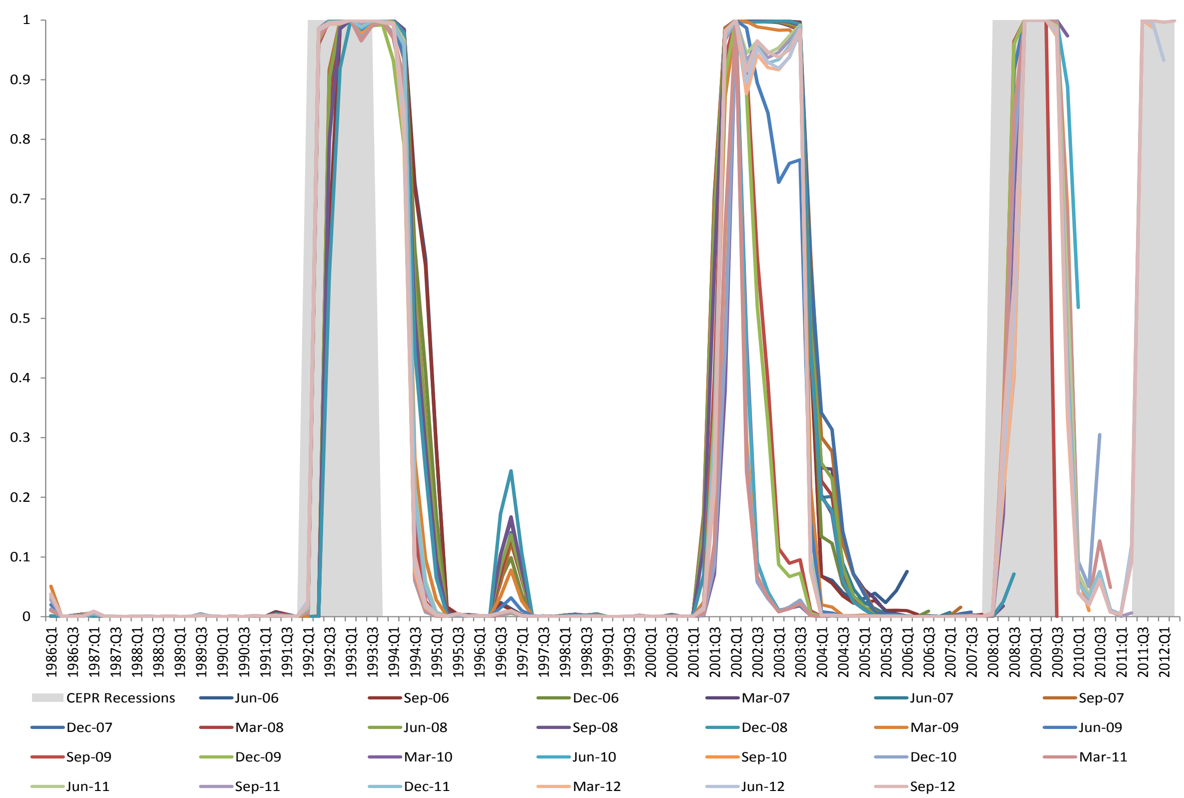
Note: This figure reports the quarterly real-time probability of a euro area recession obtained from a bivariate MS model with quarterly GDP and quarterly ESI. The probabilities are from the recursive forecasting exercise using real-time data vintages. For example, the June 2006 data vintage uses information up to 2006:Q1 for GDP and March 2006 for ESI.

Figure 7: Euro area probability of recession obtained from the recursive forecasting exercise - MSMF-VAR (SV-U) model)



Note: This figure reports the quarterly real-time probability of a euro area recession obtained from a MSMF-VAR (SV-U) model with quarterly GDP and quarterly ESI. The probabilities are from the recursive forecasting exercise using real-time data vintages. For example, the June 2006 data vintage uses information up to 2006:Q1 for GDP and March 2006 for ESI.

Figure 8: Euro area probability of recession obtained from the recursive forecasting exercise - MSMF-VAR (SV-R) model)



Note: This figure reports the quarterly real-time probability of a euro area recession obtained from a MSMF-VAR (SV-R) model with quarterly GDP and quarterly ESI. The probabilities are from the recursive forecasting exercise using real-time data vintages. For example, the June 2006 data vintage uses information up to 2006:Q1 for GDP and March 2006 for ESI.